APPLIED GENERAL TOPOLOGY

Almost periodic points and minimal sets in topological spaces

Chikara Fujita and Hisao Kato

ABSTRACT. In the paper [Almost periodic points and minimal sets in ω-regular spaces, Topology Appl. **154** (2007), 2873–2879], Mai and Sun showed that several known results concerning almost periodic points and minimal sets of maps can be generalized from regular spaces to ω-regular spaces. Also, they have three unsolved problems. In this paper, we answer to all problems which remain unsolved in the paper of Mai and Sun. In fact we prove some general theorems which give counter examples of the problems[1].

2000 AMS Classification: Primary 54H20; Secondary 54D10, 37B20, 37B35.

Keywords: Almost periodic points, minimal sets, ω-regular space.

1. Introduction

In this section, we need the following terminology and concepts. Let $\mathbb{N} = \{0, 1, 2, ...,\}$ be the set of natural numbers and let $\mathbb{Z} = \{0, \pm 1, \pm 2, ...,\}$ be the set of integers. For a set A, $|A|$ denotes the cardinality of the set A. If $f : X \to X$ is a map (=continuous function) of a topological space X, then $f^0 = Id$ and f^n $(n \geq 1)$ denotes the composition with itself n times. The orbit of a point $x \in X$ under f, denoted by $O^+(x, f)$, is the set $\{f^n(x) | \ n \in \mathbb{N}\}$. Also If $f : X \to X$ is a homeomorphism, then we put $f^{-n} = (f^{-1})^n$ $(n \geq 1)$, where f^{-1} is the inverse of f. The two-sided orbit of a point $x \in X$ under f, denoted by $O^\pm(x, f)$, is the set $\{f^n(x) | \ n \in \mathbb{Z}\}$. A point $x \in X$ is called a *periodic point* of f if there exists a positive number $N \in \mathbb{N}$ such that $f^N(x) = x$. A point $x \in X$ is called an *almost periodic point* of f provided that for any neighborhood U of x in X, there exists $N \in \mathbb{N}$ such that $\{f^{n+i}(x) | \ i = 0, 1, 2, ..., N\} \bigcap U \neq \phi$ for all $n \in \mathbb{N}$. We denote the set of all almost periodic points of f by $AP(f)$. A subset W of X is *invariant* of f if $W \neq \phi$ and $f(W) \subseteq W$. A subset W of X is a *minimal set* of f if W is a closed invariant set of f and W does not contain

---

[1]In [6], Fedeli and Le Donne gave counter examples of the problems by different methods.

any proper closed invariant set of f. A map $f : X \to X$ is *minimal* if X is a minimal set of f. It is well known that if $f : S^1 \to S^1$ is an irrational rotation of the unit circle S^1, then f is minimal.

A topological space X is a T_1-space if for any distinct points x and y in X, there exist open sets U and V of X such that $x \in U$, $y \in V$, $y \notin U$ and $x \notin V$. A topological space X is a *Hausdorff space* if for any distinct points x and y in X, there exist disjoint open sets U and V of X such that $x \in U$ and $y \in V$. A topological space X is a *regular space* if for any closed subset W of X, any point $x \in X - W$, there exist disjoint open sets U and V such that $x \in U$ and $W \subseteq V$. A topological space X is an *ω-regular space* if for any closed subset W of X, any point $x \in X - W$ and any countable subset A of W, there exist disjoint open sets U and V such that $x \in U$ and $A \subseteq V$.

The following theorem is well known (see [1], [2] and [3]).

Theorem 1.1. *Let X be a compact Hausdorff space and let $f : X \to X$ be a map. Then the followings hold.*

    (1) *If $x \in X$ is any almost periodic point of f, then $Cl(O^+(x, f))$ is a minimal set of f.*

    (2) *All points in any minimal set of f are almost periodic points.*

In [5], Mai and Sun showed that several results related to Theorem 1.1 can be generalized from regular spaces to ω-regular spaces. In this paper, we answer to all problems which remain unsolved in the paper of Mai and Sun. Consequently, we know that currently "*ω-regular space*" is the best concept in generalization.

2. Closure of almost periodic orbits in Hausdorff spaces

In [5], Mai and Sun proved that if X is an ω-regular space and $f : X \to X$ is a map, then the closure of every almost periodic orbit of f is a minimal set of f. Related to this result, they have the following problem (Problem 2.5 of [5]).

Problem 2.1. *Is the closure of an almost periodic orbit in a Hausdorff space a minimal set ?*

By use of Zorn's Lemma, we see that if X is any compact Hausdorff space and $f : X \to X$ is a map, then there is a minimal set of f. Also, there are many kinds of minimal homeomorphisms $f : X \to X$ of some uncountable compact metric spaces X. We have the following theorem which gives a counter example of this problem.

Theorem 2.2. *Suppose that $(X, \mathcal{T})$ is an uncountable compact Hausdorff space and $f : (X, \mathcal{T}) \to (X, \mathcal{T})$ is a minimal homeomorphism. Let Λ_i $(i = 1, 2)$ be any disjoint nonempty sets with $|\Lambda_1| + |\Lambda_2| = |X|$. Then there exists a topology $\mathcal{T}_\mathcal{B}$ of X and points $a_\lambda \in X$ $(\lambda \in \Lambda_1), b_\lambda \in X$ $(\lambda \in \Lambda_2)$ such that*

    (1) *$(X, \mathcal{T}_\mathcal{B})$ is a Hausdorff space with $\mathcal{T} \subseteq \mathcal{T}_\mathcal{B}$,*

    (2) *$f : (X, \mathcal{T}_\mathcal{B}) \to (X, \mathcal{T}_\mathcal{B})$ is a homeomorphism with $X = AP(f)$,*

(3) *the family* $\{O^{\pm}(a_\lambda, f) | \ \lambda \in \Lambda_1\} \bigcup \{O^{\pm}(b_\lambda, f) | \ \lambda \in \Lambda_2\}$ *is a decomposition of* X,

(4) $Cl(O^+(a_\lambda, f)) = X \ (\lambda \in \Lambda_1)$, $O^{\pm}(b_\lambda, f)$ *is a closed set of* $(X, \mathcal{T_B})$ *and hence* $Cl(O^{\pm}(b_\lambda, f)) \neq X \ (\lambda \in \Lambda_2)$, *which implies that* $f : (X, \mathcal{T_B}) \to (X, \mathcal{T_B})$ *is not minimal, and*

(5) *if* $\lambda \in \Lambda_2$ *and* $z \in X \setminus O^{\pm}(b_\lambda, f)$, *then* z *and the closed set* $O^{\pm}(b_\lambda, f)$ *can not be separated by any two open disjoint sets, in particular* $(X, \mathcal{T_B})$ *is not an* ω-*regular space.*

Proof. Note that for any $x_1, x_2 \in X$, $O^{\pm}(x_1, f) \bigcap O^{\pm}(x_2, f) \neq \phi$ if and only if $O^{\pm}(x_1, f) = O^{\pm}(x_2, f)$. Since each $O^{\pm}(x, f)$ is a countable set, we can choose $a_\lambda \in X \ (\lambda \in \Lambda_1), b_\lambda \in X \ (\lambda \in \Lambda_2)$ such that the family

$$\{O^{\pm}(a_\lambda, f) | \ \lambda \in \Lambda_1\} \bigcup \{O^{\pm}(b_\lambda, f) | \ \lambda \in \Lambda_2\}$$

is a decomposition of X. We put $K_\lambda = O^{\pm}(b_\lambda, f) \ (\lambda \in \Lambda_2)$. We consider the topology $\mathcal{T}_B$ on X as follows: For $x \notin \bigcup_{\lambda \in \Lambda_2} K_\lambda$, we consider that x has the (open) neighborhood base

$$\mathcal{B}(x) = \{U \setminus K_M | \ x \in U \in \mathcal{T}, \ K_M = \bigcup_{\lambda \in M} K_\lambda, \ M \subseteq \Lambda_2\}.$$

For $x \in K_{\lambda(x)} \ (\lambda(x) \in \Lambda_2)$, we consider that x has the neighborhood base

$$\mathcal{B}(x) = \{U \setminus K_M | \ x \in U \in \mathcal{T}, \ K_M = \bigcup_{\lambda \in M} K_\lambda, \ M \subseteq \Lambda_2 \setminus \{\lambda(x)\}\}.$$

In fact, the family $\{\mathcal{B}(x) | \ x \in X\}$ satisfies the following properties.

(1) If $B \in \mathcal{B}(x)$, then $x \in B$.
(2) If $B_1, B_2 \in \mathcal{B}(x)$, then there is $B \in \mathcal{B}(x)$ such that $B \subseteq B_1 \bigcap B_2$.
(3) If $B \in \mathcal{B}(x)$ and $x' \in B$, there is $B' \in \mathcal{B}(x')$ such that $B' \subseteq B$.

Hence we obtain the topology $\mathcal{T}_B$ on X from the neighborhood bases $\mathcal{B}(x) \ (x \in X)$. In the definition of $\mathcal{B}(x)$, if we consider the case $M = \phi$, we see that $\mathcal{T} \subseteq \mathcal{T_B}$. Hence $(X, \mathcal{T}_B)$ is a Hausdorff space such that $K_\lambda = O^{\pm}(b_\lambda, f)$ is a closed set for $\lambda \in \Lambda_2$. Let $z \in X \setminus K_\lambda$. Suppose, on the contrary, that there exist open disjoint sets W_1 and W_2 of $(X, \mathcal{T_B})$ such that $z \in W_1, K_\lambda \subseteq W_2$. We may assume that $W_1 = U \setminus K_M$, where $z \in U \in \mathcal{T}$, $K_M = \bigcup_{\lambda \in M} K_\lambda$ for some $M \subseteq \Lambda_2$. Since $f : (X, \mathcal{T}) \to (X, \mathcal{T})$ is minimal, there is a point $b \in O^+(b_\lambda, f) \subseteq K_\lambda$ such that $b \in U$. Take an open neighborhood $(U' \setminus K_{M'})$ of b in $(X, \mathcal{T_B})$ such that $U' \subseteq U$ and $(U' \setminus K_{M'}) \subseteq W_2$. Choose a point $a_{\lambda_1} \in X \ (\lambda_1 \in \Lambda_1)$. Then we have a point $a \in O^+(a_{\lambda_1}, f)$ such that $a \in U'$. We see that $a \in (U \setminus K_M) \bigcap (U' \setminus K_{M'}) \subseteq W_1 \bigcap W_2$. This is a contradiction. Since $f(K_\lambda) = K_\lambda \ (\lambda \in \Lambda_2)$, we can easily prove that $f : (X, \mathcal{T_B}) \to (X, \mathcal{T_B})$ is continuous, and hence $f : (X, \mathcal{T_B}) \to (X, \mathcal{T_B})$ is homeomorphism. To avoid confusing, we express the homeomorphism $f : (X, \mathcal{T_B}) \to (X, \mathcal{T_B})$ by $f_{\mathcal{B}}$. Since $f : (X, \mathcal{T}) \to (X, \mathcal{T})$ is minimal, by Theorem 1.1 we see $X = AP(f)$. By the definition of the neighborhood bases $\mathcal{B}(x) \ (x \in X)$, we can easily prove that

$X = AP(f_{\mathcal{B}})$. Similarly, by use of the fact $Cl(O^+(a_\lambda, f)) = X$, we can prove that for each $\lambda \in \Lambda_1$, $Cl(O^+(a_\lambda, f_{\mathcal{B}})) = X$. $\square$

3. Minimal sets in locally compact T_1-spaces

In [5], Mai and Sun proved that if X is a locally compact Hausdorff space and $f : X \to X$ is a map, then each minimal set of f is compact. Related to this result, they have the following problem (Problem 3.10 of [5]).

Problem 3.1. *Let $f : X \to X$ be a map of a locally compact T_1-space X. Is each minimal set of f compact?*

We have the following theorem which gives a counter example of this problem.

Theorem 3.2. *There exists a locally compact T_1-space Z and a map $f : Z \to Z$ such that*

(1) *Z is a minimal set of f,*
(2) *$Z = AP(f)$, and*
(3) *Z is not compact.*

Proof. Let X and Y be arbitrary infinite sets. Consider the cofinite topology $\mathcal{T}_X$ on X, i.e.,

$$\mathcal{T}_X = \{U|\ U \subseteq X \text{ and } |X \setminus U| < \infty\} \bigcup \{\phi\}.$$

It is well known that $(X, \mathcal{T}_X)$ is a compact T_1-space. Let $Z = X \bigcup Y$, where $X \bigcap Y = \phi$. We consider the following topology $\mathcal{T}_Z$ on Z. For $z \in X \subseteq Z$, z has the neighborhood base $\mathcal{B}(z) = \{U|\ z \in U \in \mathcal{T}_X\}$. For $z \in Y$, we consider that z has the neighborhood base $\mathcal{B}(z) = \{\{z\} \bigcup U|\ U \neq \phi\ \text{ and } U \in \mathcal{T}_X\}$. In fact, the family $\{\mathcal{B}(z)|\ z \in Z\}$ satisfies the following properties.

(1) If $V \in \mathcal{B}(z)$, then $z \in V$.
(2) If $V_1, V_2 \in \mathcal{B}(z)$, then there is $V \in \mathcal{B}(z)$ such that $V \subseteq V_1 \bigcap V_2$.
(3) If $V \in \mathcal{B}(z)$ and $z' \in V$, there is $W \in \mathcal{B}(z')$ such that $W \subseteq V$.

Hence we obtain the topology $\mathcal{T}_Z$ on Z from the neighborhood bases $\mathcal{B}(z)$ ($z \in Z$). Note that $(Z, \mathcal{T}_Z)$ is a locally compact T_1-space and it is not compact, because that Y is an infinite set. We can easily see that if $z \in Y$ and U is a neighborhood of z in $(Z, \mathcal{T}_Z)$, then $U \bigcap X \neq \phi$. Take an arbitrary function $g : X \to X$ such that g has no periodic point and $|g^{-1}(x)|$ is finite for each $x \in X$. Since g is a finite-to-one function, we see that $g : (X, \mathcal{T}_X) \to (X, \mathcal{T}_X)$ is continuous. Define a map $f : Z \to Z$ by $f(z) = g(z)$ for each $z \in X$, and $f(z) = h(z)$ for each $z \in Y$, where $h : Y \to X$ is an arbitrary function. Then we can easily see that $f : (Z, \mathcal{T}_Z) \to (Z, \mathcal{T}_Z)$ is continuous. Since g has no periodic point, we see that $Cl(O^+(z, f)) = Z$ for each $z \in Z$ and $Z = AP(f)$. Note that $Z = (Z, \mathcal{T}_Z)$ is a (unique) minimal set of f and Z is a locally compact T_1-space, but Z is not compact. $\square$

4. Points in closures of almost periodic points in Hausdorff spaces

In [5], Mai and Sun proved that if X is an ω-regular space and $f : X \to X$ is a map, then all points in the closure of any almost periodic orbit of f are almost periodic. Related to this result, they have the following problem (Problem 4.4 of [5]).

Problem 4.1. *Does the closure of any almost periodic orbit in a Hausdorff space contain only almost periodic points?*

We have the following theorem which gives a counter example of this problem.

Theorem 4.2. *Suppose that $(X, \mathcal{T})$ is an uncountable compact Hausdorff space and $f : (X, \mathcal{T}) \to (X, \mathcal{T})$ is a minimal homeomorphism. Let Λ_i $(i = 1, 2)$ be any disjoint nonempty sets with $|\Lambda_1| + |\Lambda_2| = |X|$. Then there exists a topology $\mathcal{T}_\mathcal{B}$ of X and points $a_\lambda \in X$ $(\lambda \in \Lambda_1), b_\lambda \in X$ $(\lambda \in \Lambda_2)$ such that*

(1) *$(X, \mathcal{T}_\mathcal{B})$ is a Hausdorff space with $\mathcal{T} \subseteq \mathcal{T}_\mathcal{B}$,*
(2) *$f = f_{\mathcal{T}_\mathcal{B}} : (X, \mathcal{T}_\mathcal{B}) \to (X, \mathcal{T}_\mathcal{B})$ is a homeomorphism,*
(3) *the family $\{O^\pm(a_\lambda, f)| \ \lambda \in \Lambda_1\} \bigcup \{O^\pm(b_\lambda, f)| \ \lambda \in \Lambda_2\}$ is a decomposition of X, and*
(4) *$O^\pm(a_\lambda, f) \subseteq AP(f)$ and $Cl(O^+(a_\lambda, f)) = X$ for $\lambda \in \Lambda_1$,*
(5) *$O^\pm(b_\lambda, f))$ is a discrete closed set of $(X, \mathcal{T}_\mathcal{B})$ and $O^\pm(b_\lambda, f) \bigcap AP(f) = \phi$ for $\lambda \in \Lambda_2$,*
(6) *if $\lambda \in \Lambda_2$ and $z \in X \setminus O^\pm(b_\lambda, f)$, then z and the closed set $O^\pm(b_\lambda, f)$ can not be separated by any two open disjoint sets, in particular $(X, \mathcal{T}_\mathcal{B})$ is not an ω-regular space.*

Proof. As in the proof of Theorem 2.2, we obtain points $a_\lambda \in X$ $(\lambda \in \Lambda_1), b_\lambda \in X$ $(\lambda \in \Lambda_2)$ such that the family

$$\{O^\pm(a_\lambda, f)| \ \lambda \in \Lambda_1\} \bigcup \{O^\pm(b_\lambda, f)| \ \lambda \in \Lambda_2\}$$

is a decomposition of X. We put $K_\lambda = O^\pm(b_\lambda, f)$ $(\lambda \in \Lambda_2)$. We consider the topology $\mathcal{T}_B$ on X as follows: For $x \notin \bigcup_{\lambda \in \Lambda_2} K_\lambda$, we consider that x has the (open) neighborhood base

$$\mathcal{B}(x) = \{U \setminus K_M| \ x \in U \in \mathcal{T}, \ K_M = \bigcup_{\lambda \in M} K_\lambda, \ M \subseteq \Lambda_2\}.$$

For $x \in K_{\lambda(x)}$ $(\lambda(x) \in \Lambda_2)$, we consider that x has the neighborhood base

$$\mathcal{B}(x) = \{(U \setminus K_M) \bigcup \{x\}| \ x \in U \in \mathcal{T}, \ K_M = \bigcup_{\lambda \in M} K_\lambda, \ \lambda(x) \in M \subseteq \Lambda_2\}.$$

In fact, the family $\{\mathcal{B}(x)| \ x \in X\}$ satisfies the following properties.

(1) If $B \in \mathcal{B}(x)$, then $x \in B$.
(2) If $B_1, B_2 \in \mathcal{B}(x)$, then there is $B \in \mathcal{B}(x)$ such that $B \subseteq B_1 \bigcap B_2$.
(3) If $B \in \mathcal{B}(x)$ and $x' \in B$, there is $B' \in \mathcal{B}(x')$ such that $B' \subseteq B$.

Hence we obtain the topology $\mathcal{T}_B$ on X from the neighborhood bases $\mathcal{B}(x)$ ($x \in X$). By the definition of $\mathcal{B}(x)$, we see that $\mathcal{T} \subseteq \mathcal{T}_\mathcal{B}$. Hence $(X, \mathcal{T}_B)$ is a Hausdorff space. Also, by the definition of $\mathcal{B}(x)$, $K_\lambda = O^\pm(b_\lambda, f)$ is a discrete closed set for $\lambda \in \Lambda_2$. Since f has no periodic point, we see that $O^\pm(b_\lambda, f) \bigcap AP(f) = \phi$ for $\lambda \in \Lambda_2$. As in the proof of Theorem 2.2, we see that $O^\pm(a_\lambda, f) \subseteq AP(f)$ and $Cl(O^+(a_\lambda, f)) = X$ for $\lambda \in \Lambda_1$. Also we see that each point $z \in X \setminus O^\pm(b_\lambda, f)$ and the closed set $O^\pm(b_\lambda, f)$ can not be separated by any two open disjoint sets, in particular $(X, \mathcal{T}_\mathcal{B})$ is not an ω-regular space. This completes the proof. $\quad \square$

References

[1] G. D. Birkhoff, Dynamical Systems, revised editions, Amer. Math. Soc. Colloq. Publ., vol. IX, Amer. Math. Soc., Providence, RI, 1966.

[2] W. H. Gottschalk, *Orbit-closure decompositions and almost periodic properties*, Bull. Amer. Math. Soc. **50** (1944), 915–919.

[3] W. H. Gottschalk, *Almost periodic points with respect to transformation semi-groups*, Ann. Math. **47** (1946), 762–766.

[4] J. L. Kelley, General Topology, Springer-Verlag, New York, 1975.

[5] J.-H. Mai and W.-H. Sun, *Almost periodic points and minimal sets in ω-regular spaces*, Topology Appl. **154** (2007), 2873–2879.

[6] A. Fedeli and A. Le Donne, *On almost periodic orbits and minimal sets*, Topology Appl. **156** (2008), no. 2, 473–475.

Chikara Fujita (chikara@math.tsukuba.ac.jp)
Institute of Mathematics, University of Tsukuba, Ibaraki, 305-8571 Japan.

Hisao Kato (hisakato@sakura.cc.tsukuba.ac.jp)
Institute of Mathematics, University of Tsukuba, Ibaraki, 305-8571 Japan.