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On Banach fixed point theorems for partial metric spaces

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Abstract. In this paper we prove several generalizations of the Banach fixed point theorem for partial metric spaces (in the sense of O'Neill) given in [14], obtaining as a particular case of our results the Banach fixed point theorem of Matthews ([12]), and some well-known classical fixed point theorems when the partial metric is, in fact, a metric.

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1. Introduction and preliminaries

In recent years many works on domain theory have been made in order to equip semantics domain with a notion of distance. In particular, Matthews ([12]) introduced the notion of a partial metric space as a part of the study of denotational semantics of dataflow networks, showing that the Banach contraction mapping theorem can be generalised to the partial metric context for applications in program verification. The existence of several connections between partial metrics and topological aspects of domain theory have been lately pointed by other authors as O'Neill ([15]), Bukatin and Scott ([3], [4]), Waszkiewicz ([23], [24]), Schellekens ([20], [21]), Escardo ([8]), Matthews ([12]) and Romaguera and Schellekens ([19], [18]).

Throughout this paper the letters $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{N}$ will denote the set of real numbers, of nonnegative real numbers and natural numbers, respectively.

Le us recall that a *partial metric* on a (nonempty) set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(i) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$;

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(*ii*) $p(x,x) \leq p(x,y)$;
(*iii*) $p(x,y) = p(y,x)$;
(*iv*) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

A *partial metric space* is a pair (X,p) such that X is a nonempty set and p is a partial metric on X.

As we indicated above, O'Neill ([15]) studied several connections between valuations and partial metrics so that he proposed one significant change to Matthews' definition of the partial metrics, such a change consists of extending their range from $\mathbb{R}^+$ to $\mathbb{R}$. According to [14], the partial metrics in the O'Neill sense will be called *dualistic partial metrics* and a pair (X,p) such that X is a (nonempty) set and p is a dualistic partial metric on X will be called a *dualistic partial metric space.*

A paradigmatic example of a dualistic partial metric space is the pair $(\mathbb{R},p)$, where $p(x,y) = x \vee y$ for all $x,y \in \mathbb{R}$.

Each dualistic partial metric p on X generates a T_0 topology $\mathcal{T}(p)$ on X which has as a base the family of open p-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

From this fact it immediately follows that a sequence $(x_n)_{n\in\mathbb{N}}$ in a dualistic partial metric space (X,p) converges to a point $x \in X$ if and only if $p(x,x) = lim_{n\to\infty} p(x,x_n)$.

Following [15] (compare [12]), a sequence $(x_n)_{n\in\mathbb{N}}$ in a dualistic partial metric space (X,p) is called a *Cauchy sequence* if there exists $lim_{n,m\to\infty} p(x_n,x_m)$.

A dualistic partial metric space (X,p) is said to be *complete* if every Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in X converges, with respect to $\mathcal{T}(p)$, to a point $x \in X$ such that $p(x,x) = lim_{n,m\to\infty} p(x_n,x_m)$.

As usual a function $f : \mathbb{R} \to \mathbb{R}$ is *monotone non-decreasing* (or monotone or increasing) if $x \geq y$ implies $f(x) \geq f(y)$.

Recently, an extension of the Banach contraction mapping theorem have been proved in the dualistic partial metric context (see [14]) in such a way that the Matthews contraction mapping theorem can be deduced as a special case of such a result. In this paper we extend the notion of contraction between partial metric spaces to the dualistic partial metric context, and present two generalizations and one "local version" of the Banach fixed point theorem given in ([14]), which recuperate the several well-known classical fixed point theorems when the dualistic partial metric converts into a metric.

2. Contractions on dualistic partial metric spaces

For the following discussion we recall some basics correspondences between dualistic partial metrics and quasi-metric spaces.

Our basic references for quasi-metric spaces are [9] and [11].

In our context by a *quasi-metric* on a set X we mean a nonnegative real-valued function d on $X \times X$ such that for all $x,y,z \in X$:
(*i*) $d(x,y) = d(y,x) = 0 \Leftrightarrow x = y$
(*ii*) $d(x,y) \leq d(x,z) + d(z,y)$.

A *quasi-metric space* is a pair (X, d) such that X is a (nonempty) set and d is a quasi-metric on X.

Each quasi-metric d on X generates a T_0-topology $\mathcal{T}(d)$ on X which has as a base the family of open d-balls $\{B_d(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_d(x,\varepsilon) = \{y \in X : d(x,y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

If d is a quasi-metric on X, then the function d^s defined on $X \times X$ by $d^s(x,y) = \max\{d(x,y), d(y,x)\}$, is a metric on X.

Next we prove a local version of the Banach fixed point theorem for dualistic partial metric spaces (Theorem 2.3 below) obtaining as a particular case of our result the classical local version of Banach's fixed point theorem for metric spaces (see [1] and [7]).

The proof of the following auxiliary results can be found in [14] (compare [12], [15] and [13]).

Lemma 2.1. *If (X,p) is a dualistic partial metric space, then the function $d_p : X \times X \to \mathbb{R}^+$ defined by $d_p(x,y) = p(x,y) - p(x,x)$ is a quasi-metric on X such that $\mathcal{T}(p) = \mathcal{T}(d_p)$.*

As a consequence of Lemma 2.1 a mapping between dualistic partial metric spaces (X,p) and (Y,q) is continuous if it is continuous between the associated quasi-metic spaces.

Lemma 2.2. *Let (X,p) be a dualistic partial metric space. Then the following assertions are equivalent:*

(1) *(X,p) is complete*
(2) *The induced metric space $(X,(d_p)^s)$ is complete.*

Furthermore $lim_{n\to\infty}(d_p)^s(a,x_n) = 0$ if and only if $p(a,a) = lim_{n\to\infty}p(a,x_n) = lim_{n,m\to\infty}p(x_n,x_m)$.

In [14] it was proved the following Banach fixed point theorem which generalies the fixed point theorem for partial metrics of Matthews given in [12].

Theorem 2.3 (Banach fixed point theorem)**.** *Let f be a mapping from a complete dualistic partial metric space (X,p) into itself such that there is a real number c with $0 \le c < 1$, satisfying*

$$(*) \qquad |p(f(x),f(y))| \le c\,|p(x,y)|\,,$$

for all $x,y \in X$. Then f has a unique fixed point.

Corollary 2.4 (Matthews)**.** *Let f be a mapping from a complete partial metric space (X,p) into itself such that there is a real number c with $0 \le c < 1$, satisfying*

$$p(f(x),f(y)) \le cp(x,y),$$

for all $x,y \in X$. Then f has a unique fixed point.

On the other hand, it was showed in [14] that the contractive condition in the statement of Theorem 2.3 can not be replaced by the contractive condition of Corollary 2.4.

In the sequel a mapping defined from a dualistic partial metric space into itself satisfying the condition (*) will be called a *contraction* with *contraction constant* c.

The following well-known local version of the Banach fixed point theorem (Proposition 2.5 below) is a useful result with many applications. In particular it allows to show that the property of having a fixed point is invariant by homotopy for contractions and nonexpansive maps in metric spaces and, as consequence, it is applied to solve practice second order homogeneous Dirichlet problems ([1], [7]). Next we extend such this result to the dualistic partial metric case.

Proposition 2.5. *Let (X, d) be a complete metric space and let $x_0 \in X$ and $r > 0$. Suppose that $f : B_d(x_0, r) \to X$ is a contraction with contraction constant L such that $d(f(x_0), x_0) < (1 - L)r$. Then f has a unique fixed point in $B_d(x_0, r)$.*

Let (X, d) be a quasi-metric space. From now on by $\overline{Y}^d$ we denote the clousure of $Y \subseteq X$ with respect to $\mathcal{T}(d)$.

Remark 2.6. Observe that, in general, the set $\overline{B_p}(0, \varepsilon) = \{y \in X : p(x, y) \leq \varepsilon + p(x, x)\}$ is not closed with respect to $\mathcal{T}(p)$. Indeed, consider the dualistic partial metric space $(\mathbb{R}, p)$ where the dualistic partial metric p is defined by $p(x, y) = x \vee y$. It is clear that $\overline{B_p}(0, 1) = (-\infty, 1]$ and $\mathbb{R} \backslash \overline{B_p}(0, 1) = (1, +\infty)$. So $\overline{B_p}(0, 1)$ is not a closed set in $(\mathbb{R}, p)$.

Proposition 2.7. *Let (X, p) be a complete dualistic partial metric space, $x_0 \in X$ and $r > 0$. Suppose that $f : B_p(x_0, r) \to X$ is a contraction with contraction constant c such that*

$$|p(f(x_0), x_0)| < (1 - c)r - 2|p(x_0, x_0)| - |p(f(x_0), f(x_0))|.$$

for all $x, y \in B_p(x_0, r)$. Then f has a unique fixed point in $B_p(x_0, r)$.

Proof. It is clear that there exists r_0 with $0 \leq r_0 < r$ such that

$$|p(f(x_0), x_0)| \leq (1 - c)r_0 - 2|p(x_0, x_0)| - |p(f(x_0), f(x_0))|.$$

Next we show that $\overline{\overline{B_p}(x_0, r_0)}^{(d_p)^s} = \overline{B_p}(x_0, r_0)$. To see this let $x \in \overline{\overline{B_p}(x_0, r_0)}^{(d_p)^s}$ and $(x_n)_{n \in \mathbb{N}} \subset \overline{B_p}(x_0, r_0)$ such that $lim_{n \to \infty}(d_p)^s(x, x_n) = 0$. Then, given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ with $p(x_0, x_n) - p(x_0, x_0) \leq r_0$ and $p(x_n, x) - p(x_n, x_n) < \varepsilon$ whenever $n \geq n_0$. Now we only have to show that $p(x, x_0) - p(x_0, x_0) \leq r_0$. Indeed,

$$\begin{aligned} p(x, x_0) - p(x_0, x_0) &\leq p(x, x_n) + p(x_n, x_0) - p(x_n, x_n) - p(x_0, x_0) \\ &< \varepsilon + r_0. \end{aligned}$$

Consequently $p(x, x_0) - p(x_0, x_0) \leq r_0$ and $\overline{\overline{B_p}(x_0, r_0)}^{(d_p)^s} \subseteq \overline{B_p}(x_0, r_0)$. Whence $\overline{\overline{B_p}(x_0, r_0)}^{(d_p)^s} = \overline{B_p}(x_0, r_0)$.

We will show that actually $f : \overline{B_p}(x_0, r_0) \to \overline{B_p}(x_0, r_0)$. To this end note that

$$\begin{aligned} |p(f(x), f(x_0))| - |p(x_0, x_0)| &\leq |p(f(x), f(x_0))| - c|p(x_0, x_0)| \\ &\leq c \cdot (|p(x, x_0)| - |p(x_0, x_0)|) \\ &\leq c \cdot (p(x, x_0) - p(x_0, x_0)) \leq cr_0 \end{aligned}$$

for all $x \in \overline{B_p}(x_0, r_0)$. Hence

$$\begin{aligned} p(f(x), x_0) - p(x_0, x_0) &\leq d_p(x_0, f(x_0)) + d_p(f(x_0), f(x)) \\ &\leq |p(f(x), f(x_0))| + |p(f(x_0), x_0)| + |p(f(x_0), f(x_0))| + |p(x_0, x_0)| \\ &\leq cr_0 + (1 - c)r_0 = r_0, \end{aligned}$$

for all $x \in \overline{B_p}(x_0, r_0)$. Therefore $f(\overline{B_p}(x_0, r_0)) \subseteq \overline{B_p}(x_0, r_0)$.

Now we can apply Theorem 2.3 to deduce that f has a fixed point in $\overline{B_p}(x_0, r_0) \subset B_p(x_0, r)$.

Finally we want to show the uniqueness. Suppose that there exist $x, y \in B_p(x_0, r)$ such that $f(x) = x$, $f(y) = y$ and $x \neq y$. Since f is a contraction in $B_p(x_0, r)$ we have that $p(x, y) = p(x, x) = p(y, y) = 0$ because of

$$|p(a, b)| = |p(f(a), f(b))| \leq c|p(a, b)|$$

for all $a, b \in \{x, y\}$, which implies that $x = y$. This concludes the proof. □

Clearly Proposition 2.5 is a particular case of Proposition 2.7 because of $|p(x_0, x_0)| = |p(f(x_0), f(x_0))| = 0$ when the dualistic partial metric is, in fact, a metric.

3. Generalized Banach's fixed point theorems for complete dualistic partial metric spaces

In the last years many authors have obtained generalizations of the Banach fixed point theorem in complete metric spaces ([2], [5], [6], [10], [16], [17]) Recently, and motivated in part by the applications to Computer Science, some generalizations of a Banach fixed point theorem have been obtained for quasi-metric and partial metric spaces (see for instance [12], [14], [22]). In this section our interest is focused on giving two Banach fixed point type theorems where the contractive condition is weakened.

One way of extending the Banach theorem arises in a natural way from approximation problems, where the contractive condition depends on a distinguished real function.

Proposition 3.1 ([7]). *Let (X, d) be a complete metric space and let $d(f(x), f(y)) \leq \Phi(d(x, y))$ for all $x, y \in X$, where $\Phi : [0, \infty) \to [0, \infty)$ is any monotone non-decresing function with $lim_{n\to\infty}\Phi^n(t) = 0$ for any fixed $t > 0$. Then f has a unique fixed point.*

The following result generalizes the preceding one.

Theorem 3.2. *Let f be a mapping of a complete dualistic partial metric space (X, p) into itself such that*

$$|p(f(x), f(y))| \leq \Phi(|p(x, y)|) \text{ for all } x, y \in X,$$

where $\Phi : [0, +\infty) \to [0, +\infty)$ is any monotone non-decreasing function with $lim_{n\to\infty}\Phi^n(t) = 0$ for each fixed $t > 0$. Then f has a unique fixed point.

Proof. First note that if $t > 0$ then $\Phi(t) < t$, because if $t \leq \Phi(t)$ then $\Phi(t) \leq \Phi(\Phi(t))$ and therefore $t \leq \Phi^2(t)$. Thus, it is easy to prove, by induction, that $t \leq \Phi^n(t)$ for $n \geq 1$. It follows that $t \leq lim_{n\to\infty}\Phi^n(t) = 0$, a contradiction.

Fix $x \in X$. It is clear that for each $n \in \mathbb{N}$ we have

$$|p(f^n(x), f^n(x))| \leq \Phi^n(|p(x, x)|) \quad \text{and} \quad \left|p(f^n(x), f^{n+1}(x))\right| \leq \Phi^n(|p(x, f(x))|).$$

Let $\alpha := max\{|p(x, x)|, |p(x, f(x))|\}$. Since, by Lemma 2.1,

$$d_p(f^n(x), f^{n+1}(x)) = p(f^n(x), f^{n+1}(x)) - p(f^n(x), f^n(x)),$$

we deduce that

$$\begin{aligned} d_p(f^n(x), f^{n+1}(x)) &\leq |p(f^n(x), f^{n+1}(x))| + |p(f^n(x), f^n(x))| \\ &\leq \Phi^n(|p(x, f(x))|) + \Phi^n(|p(x, x)|) \\ &\leq 2\Phi^n(\alpha + 1). \end{aligned}$$

Now let $n, k \in \mathbb{N}$. Then

$$\begin{aligned} d_p(f^n(x), f^{n+k}(x)) &\leq d_p(f^n(x), f^{n+1}(x)) + ... + d_p(f^{n+k-1}(x), f^{n+k}(x)) \\ &\leq \Phi^n(|p(x, f(x))|) + \Phi^n(|p(x, x)|) + ... \\ &\quad ... + \Phi^{n+k-1}(|p(x, f(x))|) + \Phi^{n+k-1}(|p(x, x)|). \end{aligned}$$

Thus

$$d_p(f^n(x), f^{n+k}(x)) \leq 2\Phi^n(\alpha + 1) + ... + 2\Phi^{n+k-1}(\alpha + 1).$$

Similarly we show that

$$d_p(f^{n+1}(x), f^n(x)) \leq 2\Phi^n(\alpha + 1),$$

and thus

$$d_p(f^{n+k}(x), f^n(x)) \leq 2\Phi^{n+k-1}(\alpha + 1) + ... + 2\Phi^n(\alpha + 1).$$

Therefore $(f^n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in the metric space $(X, (d_p)^s)$, which is complete by Lemma 2.2. So there is $a \in X$ such that $lim_{n\to\infty}(d_p)^s(a, f^n(x)) = 0$ and $p(a, a) = lim_{n\to\infty}p(a, f^n(x)) = lim_{n,m\to\infty}p(f^n(x), f^m(x))$.

We want to show that a is the unique fixed point of f. First we claim that a is a fixed point of f. Clearly we have that

$$lim_{n,m\to\infty}d_p(f^n(x), f^m(x)) = 0.$$

Moreover $lim_{n\to\infty}p(f^n(x), f^n(x)) = 0$, since

$$|p(f^n(x), f^n(x))| \leq \Phi^n(|p(x, x)|) \leq \Phi^n(|p(x, x)| + 1)$$

and $lim_{n\to\infty}\Phi^n(|p(x,x)|+1) = 0$. Thus we deduce that $lim_{n,m\to\infty}p(f^n(x), f^m(x)) = 0$. Consequently by Lemma 2.2, $p(a,a) = 0$. Whence

$$|p(f(a), f(a))| \leq \Phi(|p(a,a)|) = \Phi(0) \leq \Phi(\varepsilon) < \varepsilon.$$

It follows that $|p(f(a), f(a))| = 0$, because the last inequality holds for all $\varepsilon > 0$. On the other hand, since $lim_{n\to\infty}p(a, f^n(x)) = p(a,a) = 0$ we have

$$|p(f^{n+1}(x), f(a))| \leq \Phi(|p(f^n(a), a)|) \leq \Phi(\varepsilon) < \varepsilon$$

eventually. Thus Lemma 2.2 shows that $f(a)$ is a limit point of $(f^n(x))_{n\in\mathbb{N}}$ in $(X, (d_p)^s)$. Then $a = f(a)$.

Finally we show that a is the unique fixed point of f. To this end let $b \in X$ such that $f(b) = b$ and $b \neq a$. Then $p(a,a) = p(b,b) = p(f(a), f(b)) = 0$, since otherwise we have that the following inequality holds

$$|p(x,y)| = |p(f(x), f(y))| \leq \Phi(|p(x,y)|) < |p(x,y)|$$

for any $x, y \in \{a, b\}$. The proof is concluded. □

Remark 3.3. Observe that Theorem 2.3 follows as a special case of the preceding result if we choose $\Phi(t) = ct$ with $0 \leq c < 1$.

We will say that a dualistic partial metric space (X,p) is *bounded* if there exists $k > 0$ such that $|p(x,y)| < k$ for every $x, y \in X$. Moreover, we define the *diameter* of a subset $Y \subseteq X$ as $\delta_p(Y) = sup\{|p(x,y)| : x, y \in Y\}$ if the supremum exists and $\delta_p(Y) = +\infty$ otherwise. Note that these notions coincide with the classical notions of bounded metric space and diameter of a metric space, respectively. Furthermore $\delta_p(Y) \leq \delta_{(d_p)^s}(Y) + \delta_{w_p}(Y)$, where $\delta_{w_p}(Y) = sup\{|p(x,x)| : x \in Y\}$ if it exists and $\delta_{w_p}(Y) = +\infty$ otherwise.

Another way of attempting an extension of the Banach fixed point theorem does not rely on measuring the difference between $p(f(x), f(y))$ and $p(x,y)$, but, similarly to the metric case, the required condition relies on the behaviour of the induced quasi-metric d_p. The below technical results are useful to prove the desired theorem.

Lemma 3.4. *Let (X,p) be a dualistic partial metric space and $Y \subseteq X$. Then $\delta_{(d_p)^s}(Y) \leq 4\delta_p(Y)$.*

Proof. It is easily seen that $\delta_{w_p}(Y) \leq \delta_p(Y)$. Furthermore

$$(d_p)^s(x,y) \leq d_p(x,y) + d_p(y,x) = 2p(x,y) - p(x,x) - p(y,y),$$

whence we have

$$(d_p)^s(x,y) \leq 2|p(x,y)| + |p(x,x)| + |p(y,y)|.$$

Immediately we deduce that $\delta_{(d_p)^s}(Y) \leq 4\delta_p(Y)$. □

Lemma 3.5. *Let (X,p) be a complete dualistic partial metric space and let $\varphi : X \to \mathbb{R}^+$ be an arbitrary non-negative function. Assume that*

$$\inf\{\varphi(x) + \varphi(y) : |p(x,y)| + |p(x,x)| + |p(y,y)| \geq a\} = \mu(a) > 0 \textit{ for all } a > 0.$$

Then each sequence $(x_n)_{n\in\mathbb{N}}$ *for which* $lim_{n\to\infty}\varphi(x_n) = 0$ *converges with repect to* $\mathcal{T}((d_p)^s)$ *to the same point of* X.

Proof. Let $A_n = \{x \in X : \varphi(x) \leq \varphi(x_n)\}$. Clearly A_n is a non-empty set and any finite family of such subsets has a non-empty intersection.

We show that $lim_{n\to\infty}\delta_{(d_p)^s}(A_n) = 0$. To this end, let $\varepsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $\varphi(x_n) < \frac{1}{2}\mu(\varepsilon)$ for all $n \geq n_0$. Thus, $\varphi(x)+\varphi(y) < \mu(\varepsilon)$ whenever $x, y \in A_n$ and $n \geq n_0$. Hence, by hypothesis $|p(x, y)| < \varepsilon$. Therefore $\delta_p(A_n) < \varepsilon$ for all $n \geq n_0$. Consequently $lim_{n\to\infty}\delta_p(A_n) = 0$. By Lemma 3.4 we obtain that $lim_{n\to\infty}\delta_{(d_p)^s}(A_n) = 0$, which gives $lim_{n\to\infty}\delta_{(d_p)^s}(\overline{A_n}^{(d_p)^s}) = 0$ because of $\delta_{(d_p)^s}(\overline{A_n}^{(d_p)^s}) = \delta_{(d_p)^s}(A_n)$. We conclude from Cantor's intersection theorem that there exists a unique $x \in \cap_{n=1}^{+\infty}\overline{A_n}^{(d_p)^s}$. Furthermore, $lim_{n\to\infty}(d_p)^s(x, x_n) = 0$, since $x_n \in \overline{A_n}^{(d_p)^s}$ for each $n \in \mathbb{N}$. Whence we follow that $p(x, x) = \lim_{n\to\infty} p(x, x_n) = \lim_{n\to\infty} p(x_n, x_n)$.

Note that we have actually proved that for any sequence $(x_n)_{n\in\mathbb{N}}$ such that $lim_{n\to\infty}\varphi(x_n) = 0$ there exists a limit point $x \in X$ with respect to $\mathcal{T}((d_p)^s)$.

Let $(y_n)_{n\in\mathbb{N}}$ be any other sequence such that $lim_{n\to\infty}\varphi(y_n) = 0$ and let $y \in X$ its limit point.

Since $lim_{n\to\infty}\varphi(x_n) = lim_{n\to\infty}\varphi(y_n) = 0$ we get that, given $\varepsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that $\varphi(x_n) < \frac{1}{2}\mu(\varepsilon)$, $\varphi(y_n) < \frac{1}{2}\mu(\varepsilon)$ whenever $n \geq n_1$. In consequence $|p(x_n, y_n)|+|p(x_n, x_n)|+|p(y_n, y_n)| < \varepsilon$ whenever $n \geq n_1$, because otherwise

$$\begin{aligned}\mu(\varepsilon) &= inf\{\varphi(x) + \varphi(y) : |p(x, y)| + |p(x, x)| + |p(y, y)| \geq \varepsilon\}\\ &\leq \varphi(x_n) + \varphi(y_n) < \mu(\varepsilon),\end{aligned}$$

which is a contradiction.

We show that $x = y$. Indeed,

$$\begin{aligned}(d_p)^s(x, y) &\leq (d_p)^s(x, x_n) + (d_p)^s(x_n, y_n) + (d_p)^s(y_n, y)\\ &\leq 2\varepsilon + (d_p)^s(x_n, y_n)\\ &\leq 2\varepsilon + 2|p(x_n, y_n)| + |p(x_n, x_n)| + |p(y_n, y_n)| < 4\varepsilon.\end{aligned}$$

So $x = y$, since the preceding inequality is true for every $\varepsilon > 0$. This completes the proof. □

Theorem 3.6. *Let* (X, p) *be a complete dualistic partial metric space* (X, p) *and let* $f : X \to X$ *be a continuous mapping from* $(X, (d_p)^s)$ *to* $(X, (d_p)^s)$ *such that the functions* $\varphi(x) = d_p(x, f(x))$ *and* $\psi(x) = d_p(f(x), x)$ *satisfy the following conditions:*

(1) $\inf\{\varphi(x) + \varphi(y) + \psi(x) + \psi(y) : |p(x, y)| + |p(x, x)| + |p(y, y)| \geq a\} = \mu(a) > 0$ *for all* $a > 0$
(2) $inf_{x\in X}(\varphi(x) + \psi(x)) = 0$.

Then f *has a unique fixed point.*

Proof. First we construct a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ such that $lim_{n\to\infty}\varphi(x_n) + \psi(x_n) = 0$. Let $\varepsilon > 0$ be given then there exists $x_\varepsilon \in X$ such that $\varphi(x_\varepsilon) +$

$\psi(x_\varepsilon) < \varepsilon$. Put $x_n = x_{1/n}$ for each $n \in \mathbb{N}$. Thus the sequence $(x_n)_{n\in\mathbb{N}}$ satisfies the following:

For every $\varepsilon \geq 1$, $\varphi(x_n) + \psi(x_n) < 1 \leq \varepsilon$ whenever $n > 2$.

For every $\varepsilon < 1$, there always exists $n_0 \in \mathbb{N}$ such that $\varphi(x_n)+\psi(x_n) < \frac{1}{n_0} < \varepsilon$ for all $n \geq n_0$.

It follows that $lim_{n\to\infty}\varphi(x_n) + \psi(x_n) = 0$ as we claim.

Since $lim_{n\to\infty}\varphi(x_n)+\psi(x_n) = 0$, by Lemma 3.5 there exists a unique $x \in X$ such that $lim_{n\to\infty}(d_p)^s(x, x_n) = 0$. Then

$$\begin{aligned}
\varphi(x) - \varphi(x_n) &= p(x, f(x)) - p(x, x) - p(x_n, f(x_n)) + p(x_n, x_n) \\
&< \varepsilon + p(x, f(x)) - p(x_n, f(x_n)) \\
&< 2\varepsilon + p(x_n, f(x)) - p(x_n, f(x_n)) \\
&\leq 2\varepsilon + p(f(x_n), f(x)) - p(f(x_n), f(x_n)) < 3\varepsilon.
\end{aligned}$$

Similarly is showed that $\varphi(x_n) - \varphi(x) < 3\varepsilon$ eventually. Hence, we conclude that $\varphi(x) = lim_{n\to\infty}\varphi(x_n)$.

On the other hand,

$$lim_{n\to\infty}\varphi(x_n) \leq lim_{n\to\infty}(\varphi(x_n) + \psi(x_n)) = 0.$$

Thus we have $\varphi(x) = 0$ and $p(x, f(x)) = p(x, x)$.

Furthermore,

$$\begin{aligned}
\psi(x) - \psi(x_n) &= p(x, f(x)) - p(f(x), f(x)) - p(x_n, f(x_n)) + p(f(x_n), f(x_n)) \\
&< \varepsilon + p(x, f(x)) - p(x_n, f(x_n)) \\
&< 2\varepsilon + p(x_n, f(x)) - p(x_n, f(x_n)) \\
&\leq 2\varepsilon + p(f(x_n), f(x)) - p(f(x_n), f(x_n)) < 3\varepsilon.
\end{aligned}$$

Again we similarly show that $\psi(x_n) - \psi(x) < 3\varepsilon$ eventually.

So $\psi(x) = lim_{n\to\infty}\psi(x_n) \leq lim_{n\to\infty}(\varphi(x_n) + \psi(x_n)) = 0$. Whence we deduce that $p(x, f(x)) = p(f(x), f(x)) = p(x, x)$. By condition (i) of the definition of a dualistic partial metric $x = f(x)$, and f has a fixed point.

Next we show the uniqueness. To this end we consider $y \in X$ such that $y \neq x$ and $f(y) = y$. Then we would have $\alpha := |p(x, y)|+|p(x, x)|+|p(y, y)| \neq 0$, because otherwise $p(x, y) = p(x, x) = p(y, y) = 0$ and $x = y$. It follows that

$$\begin{aligned}
0 &< \mu(\alpha) = inf\{\varphi(x) + \varphi(y) + \psi(x) + \psi(y) : |p(x, y)| + |p(x, x)| + |p(y, y)| \geq \alpha\} \\
&\leq \varphi(x) + \varphi(y) + \psi(x) + \psi(y) = 0,
\end{aligned}$$

a contradiction. So that $x = y$. □

As a consequence we obtain the following classical result which can be found in [7].

Corollary 3.7. *Let f be a mapping from a complete metric space (X, d) into itself and $\varphi : X \to \mathbb{R}^+$ the non-negative function defined by $\varphi(x) = d(x, f(x))$. Assume that*

$$\inf\{\varphi(x) + \varphi(y) : d(x, y) \geq a\} = \mu(a) > 0 \text{ for all } a > 0$$

and that $inf_{x\in X}d(x, f(x)) = 0$. *Then* f *has a unique fixed point.*

Remark 3.8. Observe that Theorem 2.3 follows from Theorem 3.6. It is easy to see that

$$\begin{aligned} p(x,y) - p(f(x), f(y)) &\leq d_p(f(x), x) + d_p(f(y), y) \\ &= \psi(x) + \psi(y), \end{aligned}$$

and

$$\begin{aligned} p(f(x), f(y)) - p(x,y) &\leq d_p(x, f(x)) + d_p(y, f(y)) \\ &= \varphi(x) + \varphi(y). \end{aligned}$$

Thus $|p(x,y) - p(f(x), f(y))| \leq \varphi(x) + \varphi(y) + \psi(x) + \psi(y)$.

Moreover

$$\begin{aligned} |p(x,x)| - |p(f(x), f(x))| &\leq |p(x,x) - p(f(x), f(x))| \\ &\leq d_p(x, f(x)) + d_p(f(x), x) \\ &= \varphi(x) + \psi(x) \end{aligned}$$

and

$$\begin{aligned} |p(y,y)| - |p(f(y), f(y))| &\leq |p(y,y) - p(f(y), f(y))| \\ &\leq d_p(y, f(y)) + d_p(f(y), y) \\ &= \varphi(y) + \psi(y). \end{aligned}$$

Now if $|p(f(x), f(y))| \leq c|p(x,y)|$ for any $0 \leq c < 1$, bearing in mind the preceding inequalities, we follow that

$$(1-c)(|p(x,y)| + |p(x,x)| + |p(y,y)|) \leq 2(\varphi(x) + \varphi(y) + \psi(x) + \psi(y)).$$

Therefore the condition (1) of Theorem 3.6 is satisfied.

Finally from the contractive condition we have, for each $x \in X$, that $lim_{n\to\infty}(d_p)^s(f^n(x), f^{n+1}(x)) = 0$ (see proof of Theorem 2.3 in [14]). Consequently $inf_{x\in X}(\varphi(x) + \psi(x)) = 0$ because

$$\begin{aligned} inf_{x\in X}(\varphi(x) + \psi(x)) &\leq \varphi(f^n(x)) + \psi(f^n(x)) \\ &= d_p(f^n(x), f^{n+1}(x)) + d_p(f^{n+1}(x), f^n(x)) \\ &\leq 2(d_p)^s(f^n(x), f^{n+1}(x)). \end{aligned}$$

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