Generalized closed sets: a unified approach

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ABSTRACT.   We investigate various classes of generalized closed
sets of a topological space in a unified way by studying the notion
of qr-closed sets.   New characterizations of some existing classes of
generalized closed sets and topological spaces are given. A new class of
generalized closed sets, the $\tau\alpha$-closed sets, and a new class of topological
spaces, βgs-spaces, are introduced.

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1. INTRODUCTION

In General Topology, the notion of a closed set is fundamental. In 1970,
Levine [15] introduced the concept of generalized closed sets in a topological
space by comparing the closure of a subset with its open supersets. Recall
that a subset A of a topological space X is *generalized closed* (briefly, *g-closed*)
if cl$A \subseteq U$ whenever $A \subseteq U$ and U is open. Note that this definition uses
both the "closure operator" and "openness" of the superset. By considering
other generalized closure operators or classes of generalized open sets, various
notions analogous to Levine's g-closed sets have been studied, refer to [7] for
more detail.

The study of generalized closed sets has produced some new separation ax-
ioms which are between T_0 and T_1, such as $T_{\frac{1}{2}}$, T_{gs} and $T_{\frac{3}{4}}$. Some of these
have been found to be useful in computer science and digital topology, see [14]
for example. Recent work by Cao, Ganster and Reilly shows that generalized
closed sets can also be used to characterize certain classes of topological spaces
and their variations, for example the class of extremally disconnected spaces
and the class of submaximal spaces, see [5] and [6]. For convenience, we define

eleven classes of generalized closed sets in Definitions 1 and 4 below, although they can be found from most of the references in this paper.

Definition 1.1. *Let X be a topological space. A subset A of X is called:*

- (i) α-closed *if* $\mathrm{cl}(\mathrm{int}(\mathrm{cl}A)) \subseteq A$;
- (ii) semi-closed *if* $\mathrm{int}(\mathrm{cl}A) \subseteq A$;
- (iii) preclosed *if* $\mathrm{cl}(\mathrm{int}A) \subseteq A$;
- (iv) β-closed *if* $\mathrm{int}(\mathrm{cl}(\mathrm{int}A)) \subseteq A$.

Definition 1.2. *Let X be a topological space. A subset A of X is called:*

- (i) α-open *if $X \smallsetminus A$ is α-closed, or equivalently, if $A \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}A))$;*
- (ii) semi-open *if $X \smallsetminus A$ is semi-closed, or equivalently, if $A \subseteq \mathrm{cl}(\mathrm{int}A)$;*
- (iii) preopen *if $X \smallsetminus A$ is preclosed, or equivalently, if $A \subseteq \mathrm{int}(\mathrm{cl}A)$;*
- (iv) β-open *if $X \smallsetminus A$ is β-closed, or equivalently, if $A \subseteq \mathrm{cl}(\mathrm{int}(\mathrm{cl}A))$.*

Recall that the collection of all α-open subsets of X is a topology on X, called the α-*topology* [18], which is finer than the original one. We denote X with its α-topology by X_α. A set $A \subseteq X$ is α-open if and only if A is semi-open and preopen [19]. Some authors use the term *semi-preopen* (*semi-preclosed*) for β-open (β-closed).

Definition 1.3. *Let X be a topological space, and suppose $A \subseteq X$:*

- (i) *the α-closure of A, denoted by $\mathrm{cl}_\alpha A$, is the smallest α-closed set containing A;*
- (ii) *the semi-closure of A, denoted by $\mathrm{cl}_s A$, is the smallest semi-closed set containing A;*
- (iii) *the preclosure of A, denoted by $\mathrm{cl}_p A$, is the smallest preclosed set containing A;*
- (iv) *the β-closure of A, denoted by $\mathrm{cl}_\beta A$, is the smallest β-closed set containing A.*

It is well-known that $\mathrm{cl}_\alpha A = A \cup \mathrm{cl}(\mathrm{int}(\mathrm{cl}A))$, $\mathrm{cl}_s A = A \cup \mathrm{int}(\mathrm{cl}A)$, $\mathrm{cl}_p A = A \cup \mathrm{cl}(\mathrm{int}A)$ and $\mathrm{cl}_\beta A = A \cup \mathrm{int}(\mathrm{cl}(\mathrm{int}A))$.

Definition 1.4. *Let X be a topological space. A subset A of X is called:*

- (i) generalized closed *(briefly, g-closed)* [15] *if* $\mathrm{cl}A \subseteq U$ *whenever $A \subseteq U$ and U is open;*
- (ii) semi-generalized closed *(briefly, sg-closed)* [3] *if* $\mathrm{cl}_s A \subseteq U$ *whenever $A \subseteq U$ and U is semi-open;*
- (iii) generalized semi-closed *(briefly, gs-closed)* [2] *if* $\mathrm{cl}_s A \subseteq U$ *whenever $A \subseteq U$ and U is open;*
- (iv) generalized α-closed *(briefly, gα-closed)* [8] *if $\mathrm{cl}_\alpha A \subseteq U$ whenever $A \subseteq U$ and U is α-open, or equivalently, if A is g-closed with respect to the α-topology;*
- (v) α-generalized closed *(briefly, αg-closed)* [16] *if* $\mathrm{cl}_\alpha A \subseteq U$ *whenever $A \subseteq U$ and U is open;*
- (vi) gp-closed [17] *if* $\mathrm{cl}_p A \subseteq U$ *whenever $A \subseteq U$ and U is open;*

(vii) gsp-closed [9] *if* $\mathrm{cl}_\beta A \subseteq U$ *whenever* $A \subseteq U$ *and* U *is open.*

A subset A of X is *g-open* [15] (*sg-open* [3]) if $X \smallsetminus A$ is *g*-closed (*sg*-closed). Other classes of generalized open sets can be defined in a similar manner. The known relationships between the types of generalized closed sets listed in Definitions 1 and 4 are summarised in Figure 1.
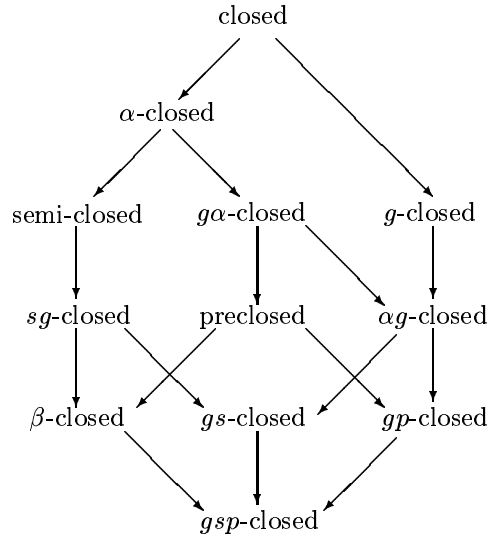

F I G U R E 1

We will address two general questions. Each generalization in Definition 1.4 involves a closure operation and a notion of "openness". Specifically, each definition involves either cl, cl_α, cl_s, cl_p, or cl_β of A together with U being either open, α-open, or semi-open. The first question, which arises from these definitions, is: do any new classes of generalized closed sets exist if we consider every possible pairing of the five closure operations mentioned above with the notions of openness in Definition 1.2? In order to study each possible pairing in a unified way, we will introduce the term *qr-closed*, where q represents a closure operation, and r represents a notion of generalized openness. Surprisingly, in most cases, we obtain new characterizations of existing classes. These cases provide new insights into the nature of generalized closed sets.

As noted above, Figure 1 summarises the known relationships between classes of generalized closed sets. In general, none of the implications represented in the diagram are reversible. The second question we will consider is: are the implications represented in the diagram the only implications which apply in general? As a consequence of answering these two questions, we will derive new relationships between different types of *qr*-closed sets which characterize certain topological spaces.

We will require the following classes of topological spaces.

Definition 1.5. *Let X be a topological space. X is:*

(i) T_{gs} [16] *if every gs-closed subset of X is sg-closed; or equivalently, if for each $x \in X$, $\{x\}$ is either closed or preopen [5];*

(ii) $T_{\frac{1}{2}}$ [15] *if every g-closed subset of X is closed; or equivalently, if for each $x \in X$, $\{x\}$ is either closed or open [12];*

(iii) semi-$T_{\frac{1}{2}}$ [3] *if every singleton is either semi-open or semi-closed in X;*

(iv) nodec [20] *if every nowhere dense set of X is closed;*

(v) nodeg *if every nowhere dense set of X is g-closed;*

(vi) extremally disconnected [5] *if the closure of each open set of X is open;*

(vii) g-submaximal [6] *if every dense subset of X is g-open;*

(viii) sg-submaximal [5] *if every dense subset of X is sg-open;*

(ix) strongly irresolvable [6] *if no open subspace is the union of two disjoint dense subsets in the subspace.*

No separation axioms are assumed unless explicitly stated.

2. A UNIFIED APPROACH: qr-CLOSED SETS

In the following we shall denote closed (resp. semi-closed, preclosed) by τ-closed (resp. s-closed, p-closed), and cl$A$ by cl$_\tau A$ for $A \subseteq X$, whenever it is convenient to do so. Similarly we denote open (resp. semi-open, preopen) by τ-open (resp. s-open, p-open). Let $\mathcal{P} = \{\tau, \alpha, s, p, \beta\}$.

Definition 2.1. *Let X be a topological space and $q, r \in \mathcal{P}$. A subset $A \subseteq X$ is called qr-closed if $\mathrm{cl}_q A \subseteq U$ whenever $A \subseteq U$ and U is r-open.*

Remark 2.2. Note that each type of generalized closed set in Definition 1.4 is defined to be qr-closed for some $q, r \in \mathcal{P}$. A set A is g-closed if it is $\tau\tau$-closed, αg-closed if it is $\alpha\tau$-closed, gs-closed if it is $s\tau$-closed, gp-closed if it is $p\tau$-closed, gsp-closed if it is $\beta\tau$-closed, $g\alpha$-closed if it is $\alpha\alpha$-closed, and sg-closed if it is ss-closed.

The proof of the following lemma is straightforward.

Lemma 2.3. *If X is a topological space, $A \subseteq X$, and $q \in \mathcal{P}$, then $x \in \mathrm{cl}_q A$ if and only if for each q-open set G, with $x \in G$, $G \cap A \neq \varnothing$.*

The following lemma gives two useful decompositions of a topological space.

Lemma 2.4. *Let X be a topological space.*

(i) [13] *Every singleton of X is either preopen or nowhere dense.*

(ii) *Every singleton of X is either open or preclosed.*

Proof. (ii) If $\{x\}$ is not open, then int$\{x\} = \varnothing$. Thus cl(int$\{x\}$) $= \varnothing$, and hence $\{x\}$ is preclosed.   □

Theorem 2.5. *Let X be a topological space. If $q, r \in \mathcal{P}$, then every qr-closed subset of X is q-closed if and only if each singleton of X is either q-open or r-closed.*

Proof. Suppose that every qr-closed set of X is q-closed. If $x \in X$ and $\{x\}$ is not r-closed, then $X \smallsetminus \{x\}$ is not r-open. Thus, the only r-open set containing $X \smallsetminus \{x\}$ is X, which implies that $X \smallsetminus \{x\}$ is qr-closed. By hypothesis, $X \smallsetminus \{x\}$ is q-closed. Therefore, $\{x\}$ is q-open.

Conversely, suppose that each singleton of X is either q-open or r-closed. Let A be qr-closed and $x \in \mathrm{cl}_q A$. If $\{x\}$ is q-open, then by Lemma 2.3, $x \in A$. If $\{x\}$ is r-closed and $x \notin A$, then $A \subseteq X \smallsetminus \{x\}$. Since A is qr-closed, we have $x \in \mathrm{cl}_q A \subseteq X \smallsetminus \{x\}$ which is a contradiction. Hence, $A = \mathrm{cl}_q A$ and A is q-closed. $\qquad\square$

Corollary 2.6. *Let X be a topological space, and let $A \subseteq X$ be a subset. If $r \in \{p, \beta\}$ then A is:*
- (i) *τr-closed if and only if it is closed;*
- (ii) *αr-closed if and only if it is α-closed;*
- (iii) *sr-closed if and only if it is semi-closed.*

Furthermore, if $r \in \{\alpha, s, p, \beta\}$ then A is:
- (iv) *pr-closed if and only if it is preclosed;*
- (v) *βr-closed if and only if it is β-closed.*

Theorem 2.7. *Let X be a topological space. Then a subset A of X is $q\alpha$-closed if and only if A is qs-closed, for any $q \in \mathcal{P}$.*

Proof. By definition, every qs-closed set is $q\alpha$-closed. To show the converse, let $A \subseteq X$ be $q\alpha$-closed and $A \subseteq U$ where U is semi-open. Suppose that there exists a point $x \in \mathrm{cl}_q A \smallsetminus U$. Then $A \subseteq U \subseteq X \smallsetminus \{x\}$. We consider the cases $\mathrm{int}(\mathrm{cl}\{x\}) = \varnothing$ and $\mathrm{int}(\mathrm{cl}\{x\}) \neq \varnothing$ separately.

If $\mathrm{int}(\mathrm{cl}\{x\}) = \varnothing$ then $\{x\}$ is α-closed and therefore $X \smallsetminus \{x\}$ is α-open. Since A is $q\alpha$-closed, we have $x \in \mathrm{cl}_q A \subseteq X \smallsetminus \{x\}$, which is a contradiction. If $\mathrm{int}(\mathrm{cl}\{x\}) \neq \varnothing$ then $x \in \mathrm{int}(\mathrm{cl}\{x\})$. Since U is semi-open, we have $\mathrm{cl}\,U = \mathrm{cl}(\mathrm{int}\,U)$. Therefore, we obtain the following:

$$\mathrm{cl}_q A \subseteq \mathrm{cl}\,A \subseteq \mathrm{cl}\,U = \mathrm{cl}(\mathrm{int}\,U) \subseteq \mathrm{cl}(\mathrm{int}(X \smallsetminus \{x\})) = X \smallsetminus \mathrm{int}(\mathrm{cl}\{x\}),$$

which gives a contradiction. $\qquad\square$

We have shown that for each $q, r \in \mathcal{P}$, the property qr-closed is equivalent to a known type of generalized closed set, except when $q = \tau$ and $r = \alpha$ (or equivalently $r = s$). The class of $\tau\alpha$-closed sets is in fact new, as we now establish. By definition, each closed set is $\tau\alpha$-closed and each $\tau\alpha$-closed set is both g-closed and $g\alpha$-closed.

Theorem 2.8. *Let X be a topological space. Then the following statements are equivalent:*
- (i) *every $\tau\alpha$-closed subset of X is closed;*
- (ii) *every $\tau\alpha$-closed subset of X is α-closed;*
- (iii) *every $\tau\alpha$-closed subset of X is semi-closed;*
- (iv) *every $g\alpha$-closed subset of X is semi-closed;*
- (v) *X is a semi-$T_{\frac{1}{2}}$ space.*

The proof of this theorem is similar to that of Theorem 2.5 so we omit it.

Theorem 2.9. *Let X be a topological space. Then every g-closed subset of X is $\tau\alpha$-closed if and only if X is a T_{gs}-space.*

Proof. Suppose that every g-closed subset of X is $\tau\alpha$-closed. Suppose that there exists a non-closed singleton $\{x\} \subseteq X$. Then $X \smallsetminus \{x\}$ is not open. Thus $X \smallsetminus \{x\}$ is g-closed. By hypothesis, $X \smallsetminus \{x\}$ is $\tau\alpha$-closed. We claim that $\{x\}$ is preopen. If not, then $\mathrm{int}(\mathrm{cl}\{x\}) = \varnothing$. So $\{x\}$ is semi-closed and $X \smallsetminus \{x\}$ is semi-open. From the above argument, $\mathrm{cl}(X \smallsetminus \{x\}) = X \smallsetminus \{x\}$. Thus, $X \smallsetminus \{x\}$ is closed and $\{x\}$ is open. This is impossible.

Conversely, let $A \subseteq X$ be g-closed and let U be semi-open with $A \subseteq U$. Suppose that there is a point $x \in \mathrm{cl}A \smallsetminus U$. Then $A \subseteq U \subseteq X \smallsetminus \{x\}$. If $\{x\}$ is closed, then $X \smallsetminus \{x\}$ is open. So $x \in \mathrm{cl}A \subseteq X \smallsetminus \{x\}$. This is a contradiction. If $\{x\}$ is preopen, then $x \in \mathrm{int}(\mathrm{cl}\{x\})$. Since U is semi-open, then $U \subseteq \mathrm{cl}(\mathrm{int}U)$. Therefore, we have the following $A \subseteq U \subseteq \mathrm{cl}(\mathrm{int}U) \subseteq X \smallsetminus \mathrm{int}(\mathrm{cl}\{x\})$. It follows that $\mathrm{cl}A \cap \mathrm{int}(\mathrm{cl}\{x\}) = \varnothing$. This again leads to a contradiction. $\qquad \square$

Theorem 2.10. *Let X be a topological space. Then the following statements are equivalent:*

  (i) *X is nodec;*
  (ii) *each α-closed subset of X is closed;*
  (iii) *each $g\alpha$-closed subset of X is $\tau\alpha$-closed;*
  (iv) *each α-closed subset of X is $\tau\alpha$-closed.*

Proof. (i) $\Rightarrow$ (ii). Let $A \subseteq X$ be α-closed. Then $A = \mathrm{cl}(\mathrm{int}(\mathrm{cl}A)) \cup N$, where $N = A \smallsetminus \mathrm{cl}(\mathrm{int}(\mathrm{cl}A)))$ is a nowhere dense subset of X. Thus, A is closed.

(ii) $\Rightarrow$ (i). Every nowhere dense subset of X is α-closed, thus closed.

(i) $\Rightarrow$ (iii). If X is a nodec space, then $\tau = \alpha$. Thus, for any subset $A \subseteq X$, we have $\mathrm{cl}A = \mathrm{cl}_\alpha A$. This implies that every $g\alpha$-closed subset is $\tau\alpha$-closed.

(iii) $\Rightarrow$ (iv) is obvious.

(iv) $\Rightarrow$ (i). Suppose that each α-closed subset of X is $\tau\alpha$-closed. Let $A \subseteq X$ be a nowhere dense set. We shall show that A is closed. If not, there exists a point $x \in \mathrm{cl}A \smallsetminus A$. Since $\mathrm{cl}A$ is also nowhere dense, $\{x\}$ must be nowhere dense as well. Thus, both A and $\{x\}$ are α-closed. By hypothesis, A is $\tau\alpha$-closed such that $A \subseteq X \smallsetminus \{x\}$. Since $X \smallsetminus \{x\}$ is α-open, we have $x \in \mathrm{cl}A \subseteq X \smallsetminus \{x\}$. This is a contradiction, which implies that A is closed. $\qquad \square$

From Theorems 2.8, 2.9 and 2.10, we see that in a general topological space $\tau\alpha$-closed sets are not equivalent to closed sets, g-closed sets, $g\alpha$-closed sets, semi-closed or α-closed sets.

3. Relationships

We now consider the completeness of Figure 1. We will introduce a new relationship not present in Figure 1, and establish that no other relationships exist in the general case. It follows from Theorem 2.7 that every $g\alpha$-closed set

is sg-closed. This implication cannot be reversed in general by the following theorem.

Theorem 3.1. *Let X be a topological space. Each sg-closed subset of X is $g\alpha$-closed if and only if X is extremally disconnected.*

Proof. Suppose that X is extremally disconnected. Let $A \subseteq X$ be sg-closed and let U be an α-open set containing A. Then $\mathrm{cl}_s A \subseteq U$, i.e. $\mathrm{int}(\mathrm{cl}A) \subseteq U$. Since $\mathrm{int}(\mathrm{cl}A)$ is closed, we have $\mathrm{cl}_\alpha A = A \cup \mathrm{cl}(\mathrm{int}(\mathrm{cl}A)) = A \cup \mathrm{int}(\mathrm{cl}A) \subseteq U$. Hence, A is $g\alpha$-closed.

To prove the converse, let every sg-closed subset of X be $g\alpha$-closed. Let $A \subseteq X$ be regular open. By definition, $A = \mathrm{int}(\mathrm{cl}A)$. Then A is semiclosed and so $g\alpha$-closed. It follows that $\mathrm{cl}A = \mathrm{cl}(\mathrm{int}(\mathrm{cl}A)) = \mathrm{cl}_\alpha A \subseteq A$. Therefore, A is closed and X is extremally disconnected. $\qquad\square$

Next we establish that no further relationships exist in general. First we confirm that in general none of the implications in Figure 1 can be reversed. With the exception of two cases, it has been shown that the reverse implications occur only if the space has a specific property [5], [6], [7], [4], [10]. Theorem 3.2 below addresses one of the remaining cases. The other generates a new topological property.

Theorem 3.2. *Let X be a topological space. Then X is nodeg if and only if every αg-closed subset of X is g-closed.*

Proof. Suppose that each αg-closed subset of X is g-closed. Since each nowhere dense set is α-closed, then it is g-closed.

Conversely, suppose that each nowhere dense subset of X is g-closed. Let $A \subseteq X$ be an αg-closed subset with $A \subseteq U$, where U is open. By assumption, $\mathrm{cl}_\alpha A = A \cup \mathrm{cl}(\mathrm{int}(\mathrm{cl}A)) \subseteq U$. Note that $N = A \smallsetminus \mathrm{cl}(\mathrm{int}(\mathrm{cl}A))$ is nowhere dense, and hence g-closed by assumption. Now $\mathrm{cl}N \subseteq U$ since $N \subseteq U$. Moreover, $X \smallsetminus \mathrm{cl}(\mathrm{int}(\mathrm{cl}A))$ is open and $N = A \cap (X \smallsetminus \mathrm{cl}(\mathrm{int}(\mathrm{cl}A)))$, so we have $\mathrm{cl}A \cap (X \smallsetminus \mathrm{cl}(\mathrm{int}(\mathrm{cl}A))) \subseteq \mathrm{cl}N \subseteq U$. It follows readily that $\mathrm{cl}A \smallsetminus \mathrm{cl}_\alpha A \subseteq U$ and so $\mathrm{cl}A \subseteq U$. Therefore, A is g-closed. $\qquad\square$

As promised we define a new class of topological spaces.

Definition 3.3. *A space X to be βgs if every gsp-closed subset of X is gs-closed.*

It is shown in [4] that X is a βgs-space if and only if every β-closed subset of X is gs-closed. The following implications follow from definitions and characterizations of g-submaximality of X_α in [6].

$$X_\alpha \text{ is } g\text{-submaximal} \to X \text{ is } \beta gs \to X \text{ is } sg\text{-submaximal}$$

Note that if X is a T_{gs}-space, the lefthand arrow is reversible; and if X is extremally disconnected, then the righthand arrow is reversible. We shall show that neither of these two arrows is reversible in general. In fact, we observe that the space X defined in Example 3.5 of [5] is sg-submaximal, but not βgs.

In the following, we shall provide two examples of finite spaces to distinguish these three classes of spaces.

Example 3.4. Let $X = \{a, b, c, d\}$, and let $\tau = \{\phi, X, \{a\}, \{c, d\}, \{a, c, d\}\}$. Then X is sg-submaximal, but it is not a βgs-space, since $\{a, c\}$ is β-closed but not gs-closed.

Example 3.5. Let $X = \{a, b, c, d, e\}$ and let

$$\mathcal{B} = \{\{b\}, \{d\}, \{a, b\}, \{d, e\}, \{b, c, d, e\}\}$$

be a base for a topology on X. Then X is a βgs-space, but X_α is not g-submaximal since $\{a, b, c, d\}$ is dense in X_α, but not g-open in X_α.

No further relationships exist in general by [5], [6], [7], [4], Theorems 2.8, 2.9, 2.10, 3.1, 3.2, and the following two theorems.

Theorem 3.6. *Let X be a topological space. Then the following statements are equivalent:*

   (i) *X is a T_{gs}-space;*
   (ii) *every g-closed subset of X is $g\alpha$-closed;*
  (iii) *every αg-closed subset of X is sg-closed;*
  (iv) *every g-closed subset of X is β-closed;*
   (v) *every g-closed subset of X is p-closed;*
  (vi) *every g-closed subset of X is sg-closed;*
 (vii) *every αg-closed subset of X is β-closed;*
(viii) *every gs-closed subset of X is β-closed.*

The proof of the preceding theorem is similar to that of Theorem 2.9 and of Theorem 3.6 of [7].

Theorem 3.7. *Let X be a topological space. Then:*

   (i) *X is extremally disconnected if and only if every semi-closed subset of X is αg-closed;*
   (ii) *X is extremally disconnected if and only if every sg-closed subset of X is αg-closed;*
  (iii) *X is nodeg and extremally disconnected if and only if every semi-closed subset of X is g-closed;*
  (iv) *X is $T_{\frac{1}{2}}$ if and only if every $g\alpha$-closed subset of X is semi-closed;*

Proof. (i) and (ii) Similar to the proof of [7, Theorem 4.2].

(iii) By Theorem 2.3 of [5], X is extremally disconnected if and only if every semi-closed set of X is α-closed. Moreover, by a similar argument to that of Theorem 3.2 (i), X is nodeg if and only if every α-closed set is g-closed. These facts combined with (i) complete the proof.

(iv) Similar to the proof of Theorem 2.5.   □

Thus we have a new diagram, Figure 2 below, showing all relationships between the classes of generalized closed sets under discussion. None of the implications shown in Figure 2 can be reversed in general topological spaces.
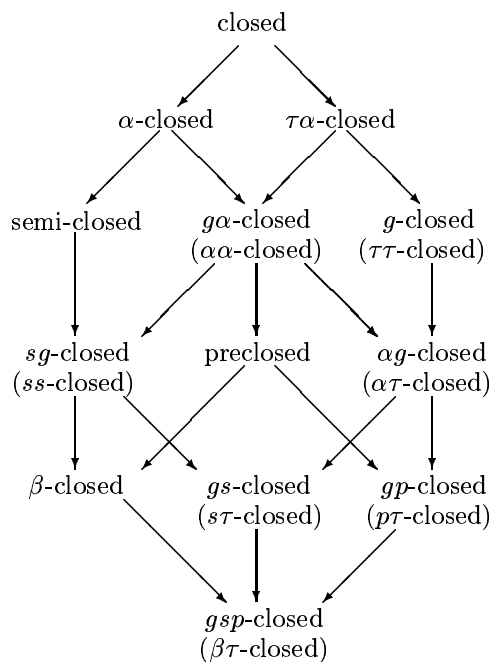
FIGURE 2

4. SUMMARY

The above results are summarised in the following table. Each cell gives the type of generalized closed set which is qr-closed, where q is given by the left-hand (zeroth) column and r is given by the top (zeroth) row.

	τ	α	s	p	β
τ	g-closed	$\tau\alpha$-closed	$\tau\alpha$-closed	closed	closed
α	αg-closed	$g\alpha$-closed	$g\alpha$-closed	α-closed	α-closed
s	gs-closed	sg-closed	sg-closed	semi-closed	semi-closed
p	gp-closed	preclosed	preclosed	preclosed	preclosed
β	gsp-closed	β-closed	β-closed	β-closed	β-closed

TABLE 1

The table highlights some general relationships between certain groups of generalized closed sets. For example column 2 implies column 1[1]. In fact each column in Table 1 implies each of the preceding columns. Each of these implications, apart from columns 3 and 4, follow immediately from the definitions, since the types of generalized closed sets in any particular column involve the

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[1]Each type of generalized closed set listed in column 2 implies the type of generalized closed set listed in the same row of column 1.

same notion of closure, and these notions of closure increase in strength from left to right. Similarly each row implies each subsequent row, apart from rows 3 and 4.

In most cases the converse relationships, between rows or columns, are equivalent to the spaces having certain properties. If X is a topological space, then the following are equivalent:

(i) X is extremally disconnected;
(ii) row 5 implies row 4 ([5], [4]);
(iii) row 3 implies row 2 (Theorem 3.1 and [5], [7]).

Row 4 implies row 3 if and only if the space is sg-submaximal, except for columns 4 and 5. This case has been considered in [1] where it is shown that X is strongly irresolvable if and only if every preclosed set is semi-closed. Therefore, as a direct consequence of Theorem 3.3 of [6], we have the following:

Theorem 4.1. *A topological space X is strongly irresolvable if and only if X is both sg-submaximal and semi-$T_{\frac{1}{2}}$.*

Row 2 implies row 1 if and only if the space is nodec (Theorem 2.10) except for the first column. In this case row 1 implies row 2 if and only if the space is nodeg (Theorem 3.2), which is a property strictly weaker than nodec.

A topological space X is T_{gs} if and only if column 1 implies column 2 ([4], [16] and Theorem 3.2). We notice that columns 2 and 3, and columns 4 and 5 are identical. Finally, column 3 implies column 4 if and only if the space is semi-$T_{\frac{1}{2}}$, except for rows 4 and 5.

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