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Fuzzy functions: a fuzzy extension of the category SET and some related categories

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Abstract. In research works where fuzzy sets are used, mostly certain usual functions are taken as morphisms. On the other hand, the aim of this paper is to fuzzify the concept of a function itself. Namely, a certain class of L-relations $F : X \times Y \to L$ is distinguished which could be considered as fuzzy functions from an L-valued set (X, E_X) to an L-valued set (Y, E_Y). We study basic properties of these functions, consider some properties of the corresponding category of L-valued sets and fuzzy functions as well as briefly describe some categories related to algebra and topology with fuzzy functions in the role of morphisms.

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1. Introduction

In research works where fuzzy sets are involved, in particular, in the theory of fuzzy topological spaces, fuzzy algebra, fuzzy measure theory, etc., mostly certain usual functions are taken as morphisms: they can be certain mappings between the corresponding sets, or between the fuzzy powersets of these sets, etc. On the other hand, there are only few papers where attempts to fuzzify the concept of a function itself are undertaken (see e.g. [11, 12], etc). The aim of our work is also to present a possible approach to this problem. Namely, a certain class of L-relations (i.e. mappings $f : X \times Y \to L$) is distinguished which seem reasonable to be viewed as (L-)*fuzzy functions* from a set X to a set Y. We define *composition* of fuzzy functions; study *images* and *preimages* of L-sets under fuzzy functions; introduce properties of *injectivity* and *surjectivity* for them; describe *products* and *coproducts* in the corresponding category, etc. In the last part of the paper we define some categories related to topology and algebra where fuzzy functions play the role of morphisms.

In conclusion, we would like to mention the following two peculiarities of our approach.

First, the appropriate context for our work is formed not by usual sets, or by their L-subsets (i.e. mappings $f : X \to L$), but rather by L-valued sets (i.e. sets, endowed with an L-valued equality $E : X \times X \to L$, see e.g. [6, 7]) and their L-subsets. And second, in the result we obtain not a usual category, but the so called *a fuzzy category* - a concept introduced and studied in [14, 15].

2. PREREQUISITES

Let $L = (L, \leq, \wedge, \vee, *)$ be an infinitely distributive GL-monoid (cf. e.g. [6], [7]), i.e. a commutative integral divisible cl-monoid (cf. [1]). It is well known that every GL-monoid is residuated, i.e. there exists a further binary operation - implication "$\longmapsto$" such that

$$\alpha * \beta \leq \gamma \Longleftrightarrow \alpha \leq \beta \longmapsto \gamma \quad \forall \alpha, \beta, \gamma \in L.$$

We set $\alpha^2 = \alpha * \alpha$ and further by induction: $\alpha^n = \alpha^{n-1} * \alpha$. Let $\top$ and $\bot$ denote respectively the top and the bottom elements of L.

Following U.Höhle (cf e.g. [7]) by an *L-valued set* we call a pair (X, E) where X is a set and E is an *L-valued equality*, i.e. a mapping $E : X \times X \to L$ such that

(1eq) $E(x,y) \leq E(x,x) \wedge E(y,y) \quad \forall x, y \in X$;
(2eq) $E(x,y) = E(y,x) \quad \forall x, y \in X$;
(3eq) $E(x,y) * \big(E(y,y) \longmapsto E(y,z)\big) \leq E(x,z) \quad \forall x, y, z \in X$.

An L-valued set (X, E) is called *separated* if

(4eq) $E(x,x) \bigvee E(y,y) \leq E(x,y) \Longleftrightarrow x = y \quad \forall x, y \in X$.

An L-valued equality E is called *global* if

(5eq) $E(x,x) = \top \quad \forall x \in X$.

Further, recall that an L-set, or more precisely, an L-subset of a set X is just a mapping $A : X \to L$. In case (X, E) is an L-valued set, its L-subset A is called *strict*, if $A(x) \leq E_X(x,x) \, \forall x \in X$; A is called *extensional* if $\sup_x A(x) * \big(E(x,x) \longmapsto E(x,x')\big) \leq A(x'), \forall x' \in X$.

By $L - SET(L)$ we denote the category whose objects are triples (X, E, A) where (X, E) is an L-valued set and A is its strict extensional L-subset, and morphisms from (X, E_X, A) to (Y, E_Y, B) are mappings $f : X \to Y$ which preserve equalities (i.e. $E_X(x_1, x_2) \leq E_Y(fx_1, fx_2)$) and "respect L-subsets", i.e. $A \leq B \circ f$. Let $L - SET'(L)$ stand for the full subcategory of the category $L - SET(L)$ determined by global separated L-valued sets.

To recall the concept of an *$L-$fuzzy category* [14, 15], consider an ordinary (classical) category $\mathcal{C}$ and let $\omega : \mathrm{Ob}(\mathcal{C}) \to L$ and $\mu : \mathrm{Mor}(\mathcal{C}) \to L$ be L-fuzzy subclasses of its objects and morphisms respectively. Now, an *L-fuzzy category* can be defined as a triple $(\mathcal{C}, \omega, \mu)$ satisfying the following axioms ([15], cf. also [14] in case $* = \wedge$):

1^0 $\mu(f) \leq \omega(X) \wedge \omega(Y) \quad \forall X, Y \in \mathrm{Ob}(\mathcal{C})$ and $\forall f \in \mathrm{Mor}(X, Y)$;
2^0 $\mu(g \circ f) \geq \mu(f) * \mu(g)$ whenever the composition $g \circ f$ is defined;

3^0 $\mu(e_X) = \omega(X)$ where $e_X : X \to X$ is the identity morphism.

Our aim is, starting from the category $L-SET(L)$, to define a fuzzy category $L-\mathcal{F}SET(L)$ having the same class of objects as $L-SET(L)$ but an essentially wider class of "potential" morphisms.

3. FUZZY CATEGORY $L - \mathcal{F}SET(L)$.

3.1. **Category $L - FSET(L)$.** We start with defining a usual (i.e. crisp) category $L - FSET(L)$. Namely, let $L - FSET(L)$ denote the category having the same objects as $L-SET(L)$ and whose morphisms, called *(potential) fuzzy functions*, from (X, E_X, A) to (Y, E_Y, B) are L-mappings $F : X \times Y \to L$ such that

(0ff) $F(x,y) \le E_X(x,x) \wedge E_Y(y,y) \quad \forall y \in Y, \forall x \in X;$
(1ff) $\sup_x A(x) * \big(E_X(x,x) \longmapsto F(x,y)\big) \le B(y) \quad \forall y \in Y;$
(2ff) $F(x,y) * \big(E_Y(y,y) \longmapsto E_Y(y,y')\big) \le F(x,y') \quad \forall x \in X, \forall y, y' \in Y;$
(3ff) $E_X(x,x') * \big(E_X(x,x) \longmapsto F(x,y)\big) \le F(x',y) \quad \forall x, x' \in X, y \in Y;$
(4ff) $F(x,y) * \big(E_X(x,x) \longmapsto F(x,y')\big) \le E_Y(y,y') \quad \forall x \in X, \forall y, y' \in Y;$

In particular, when $A = \top_X$ and $B = \top_Y$ we write $F : (X, E_X) \to (Y, E_Y)$ instead of $F : (X, E_X \top_X) \to (Y, E_Y \top_Y)$.

Notice that conditions (0ff) - (3ff) say that F is a certain L-relation, while axiom (4ff) specifies that the L-relation F is a *function*.

Remark 3.1. Since $F(x,y) \le E_X(x,x)$, and $a \le b \Longrightarrow a = b * (b \longmapsto a)$ (by divisibility of L), we have

$$\begin{aligned} & F(x,y) * (E_X(x,x) \longmapsto E_X(x,x')) \\ = \;& E_X(x,x) * (E_X(x,x) \longmapsto F(x,y)) * (E_X(x,x) \longmapsto E_X(x,x')) \\ = \;& E_X(x,x') * (E_X(x,x) \longmapsto F(x,y)). \end{aligned}$$

Therefore axiom (3ff) can be given in the following equivalent form:

(3′ff) $F(x,y) * \big(E_X(x,x) \longmapsto E_X(x,x')\big) \le F(x',y).$

Remark 3.2. Applying (4ff) it is easy to establish that

$$\begin{aligned} F(x,y_1) * F(x,y_2) &\le F(x,y_1) * \big(E_X(x,x) \longmapsto F(x,y_2)\big) \\ &\le E_Y(y_1,y_2) \\ &\le E_Y(y_1,y_1) \longmapsto E_Y(y_1,y_2). \end{aligned}$$

Remark 3.3. Let $F : (X, E_X) \to (Y, E_Y)$ be a fuzzy function, $X' \subset X$ $Y' \subset Y$, and let the L-equalities $E_{X'}$ and $E_{Y'}$ on X' and Y' be defined as the restrictions of the equalities E_X and E_Y respectively. Then defining a mapping $F' : X' \times Y' \to L$ by the equality $F'(x,y) = F(x,y) \quad \forall x \in X', \forall y \in Y'$ a fuzzy function $F' : (X', E_{X'}) \to (Y', E_{Y'})$ is obtained. We refer to it as the *restriction of* F to the subspaces $(X', E_{X'})$ and $(Y', E_{Y'})$.

Given two fuzzy functions $F : (X, E_X, A) \to (Y, E_Y, B)$ and $G : (Y, E_Y, B) \to (Z, E_Z, C)$ we define their *composition* $G \circ F : (X, E_X, A) \to (Z, E_Z, C)$ by the formula

$$(G \circ F)(x,z) = \bigvee_{y \in Y} \Big(F(x,y) * \big(E_Y(y,y) \longmapsto G(y,z)\big)\Big).$$

Since, by divisibility of L, $F(x,y) = E_Y(y,y) * \big(E_Y(y,y) \longmapsto F(x,y)\big)$ and $G(y,z) = E_Y(y,y) * \big(E_Y(y,y) \longmapsto G(y,z)\big)$, the composition can be defined also by the formula

$$(G \circ F)(x,z) = \bigvee_{y \in Y} \Big(\big(E_Y(y,y) \longmapsto F(x,y)\big) * G(y,z)\Big).$$

Proposition 3.4. *$G \circ F : (X, E_X, A) \to (Z, E_Z, C)$ is indeed a fuzzy function.*

Proof. The proof of the validity of (0ff) is straightforward.

(1ff): Taking into account divisibility of L, strictness of A and axiom (1ff) for F we get:

$$\begin{aligned}
&\sup_x \big[A(x) * \big(E_X(x,x) \longmapsto (G \circ F)(x,z)\big)\big] \\
&= \sup\nolimits_x \big(E_X(x,x) \longmapsto A(x)\big) * (G \circ F)(x,z) \\
&= \textstyle\bigvee_{x,y} \big(E_X(x,x) \longmapsto A(x)\big) * F(x,y) * \big(E_Y(y,y) \longmapsto G(y,z)\big) \\
&\le \textstyle\bigvee_{y \in Y} B(y) * \big(E_Y(y,y) \longmapsto G(y,z)\big) \\
&\le C(z).
\end{aligned}$$

(2ff): By axiom (2ff) for G we have

$$E_Z(z,z) \longmapsto E_Z(z,z') \le G(y,z) \longmapsto G(y,z') \quad \forall y \in Y, \forall z,z' \in Z.$$

Then for fixed $x \in X$, $y \in Y$ and $z, z' \in Z$ we have

$$\begin{aligned}
&F(x,y) * \big(E_Y(y,y) \longmapsto G(y,z)\big) * \big(E_Z(z,z) \longmapsto E_Z(z,z')\big) \\
&\le F(x,y) * \big(E_Y(y,y) \longmapsto G(y,z)\big) * \big(G(y,z) \longmapsto G(y,z')\big) \\
&\le F(x,y) * \big(E_Y(y,y) \longmapsto G(y,z')\big).
\end{aligned}$$

Now taking suprema by $y \in Y$ on the both sides of the inequality we get:

$$(G \circ F)(x,z) * \big(E_Z(z,z) \longmapsto E_Z(z,z')\big) \le (G \circ F)(x,z').$$

(3ff) (We prove this axiom in the form (3′ff)): Applying (3′ff) for F we have

$$\begin{aligned}
&(G \circ F)(x,z) * \big(E_X(x,x) \longmapsto E_X(x,x')\big) \\
&= \textstyle\bigvee_y F(x,y) * \big(E_Y(y,y) \longmapsto G(y,z)\big) * \big(E_X(x,x) \longmapsto E_X(x,x')\big) \\
&\le \textstyle\bigvee_y F(x',y) * \big(E_Y(y,y) \longmapsto G(y,z)\big) \\
&= (G \circ F)(x',z)
\end{aligned}$$

(4ff): We have to show that for all $x \in X$, $z, z' \in Z$

$$(G \circ F)(x,z) * \big(E_X(x,x) \longmapsto (G \circ F)(x,z')\big) \le E_Z(z,z').$$

To establish this inequality we have to show that for any $y, y' \in Y$ it holds:

$$\begin{aligned}
&\big[F(x,y) * \big(E_Y(y,y) \longmapsto G(y,z)\big)\big] * \\
&\quad \big[E_X(x,x) \longmapsto \Big(F(x,y') * \big(E_Y(y',y') \longmapsto G(y',z')\big)\Big)\big] \\
&\le E_Z(z,z').
\end{aligned}$$

By divisibility of L, axiom (4ff) for F and G and axiom (3ff) for G, we have:

$$\begin{aligned}
&\big[F(x,y) * \big(E_Y(y,y) \longmapsto G(y,z)\big)\big] * \\
&\quad \Big[E_X(x,x) \longmapsto \Big(F(x,y') * \big(E_Y(y',y') \longmapsto G(y',z')\big)\Big)\Big] \\
&\quad = \; F(x,y) * \big(E(y,y) \longmapsto G(y,z)\big) * \\
&\qquad\quad \Big(E(x,x) \longmapsto E(x,x) * \\
&\qquad\qquad \big(E(x,x) \longmapsto F(x,y')\big) * \big(E(y',y') \longmapsto G(y',z')\big)\Big) \\
&\quad = \; \big[F(x,y) * \big(E_Y(y,y) \longmapsto G(y,z)\big)\big] * \\
&\qquad\quad \big(E(x,x) \longmapsto F(x,y')\big) * \big(E(y',y') \longmapsto G(y',z')\big) \\
&\quad \leq \; F(x,y) * \big(E_Y(y,y) \longmapsto G(y,z)\big) * \\
&\qquad\quad \big(F(x,y) \longmapsto E_Y(y,y')\big) * \big(E_Y(y',y') \longmapsto G(y',z')\big) \\
&\quad \leq \; E_Y(y,y') * \big(E(y,y) \longmapsto G(y,z)\big) * \big(E(y',y') \longmapsto G(y',z')\big) \\
&\quad \leq \; G(y',z) * \big(E(y',y') \longmapsto G(y',z')\big) \\
&\quad \leq \; E(z,z').
\end{aligned}$$

By a direct verification it is easy to show that the operation of composition is associative: given fuzzy functions $F : (X, E_X, A) \to (Y, E_Y, B)$, $G : (Y, E_Y, B) \to (Z, E_Z, C)$, and $H : (Z, E_Z, C) \to (T, E_T, D)$ it holds $(H \circ G) \circ F = H \circ (G \circ F) : (X, E_X, A) \to (T, E_T, D)$. Further, the *identity morphism* is defined by the corresponding L-valued equality: $E_X : (X, E_X, A) \to (X, E_X, A)$. It is easy to verify that it satisfies the conditions (0ff) - (4ff) above and that $F \circ E_X = E_X$ and $E_Y \circ F = E_Y$ for each fuzzy function $F : (X, E_X, A) \to (Y, E_Y, B)$. Thus $L - FSET(L)$ is indeed a category. □

Remark 3.5. In case when the equalities E_X and E_Y on X and Y respectively, are global, the condition (0ff) becomes redundant and the conditions (1ff) - (4ff) can be reformulated in the following simpler way:

(1ff) $\sup_x A(x) * F(x,y) \leq B(y) \quad \forall y \in Y$;
(2ff) $F(x,y) * E_Y(y,y') \leq F(x,y') \quad \forall x \in X, \forall y, y' \in Y$;
(3ff) $E_X(x,x') * F(x,y) \leq F(x',y) \quad \forall x, x' \in X, \forall y \in Y$;
(4ff) $F(x,y) * F(x,y') \leq E_Y(y,y') \quad \forall x \in X, \forall y, y' \in Y$.

3.2. **Fuzzy category $L - \mathcal{F}SET(L)$.** Given a fuzzy function $F : (X, E_X, A) \to (Y, E_Y, B)$ let

$$\mu(F) = \inf_x \sup_y F(x,y).$$

Thus we define an L-subclass μ of the class of all morphisms of $L - FSET(L)$. In case $\mu(F) \geq \alpha$ we refer to F as a *fuzzy α-function.* If $F : (X, E_X, A) \to (Y, E_Y, B)$ and $G : (Y, E_Y, B) \to (Z, E_Z, C)$ are fuzzy functions, then $\mu(G \circ F) \geq \mu(G) * \mu(F)$. Indeed, let $x \in X$ and $y \in Y$ be fixed. Then

$$\sup_z F(x,y) * \big(E_Y(y,y) \longmapsto G(y,z)\big) \geq F(x,y) * \sup_z G(y,z) \geq F(x,y) * \mu(G),$$

and therefore for a fixed $x \in X$

$$\sup_y \sup_z F(x,y) * \big(E_Y(y,y) \longmapsto G(y,z)\big) \geq \sup_y F(x,y) * \mu(G) \geq \mu(F) * \mu(G).$$

Since $x \in X$ is arbitrary, we get $\mu(G \circ F) \geq \mu(G) * \mu(F)$.

Further, given an L-valued set (X, E) let

$$\omega(X, E) := \mu(E) = \inf_x E(x, x).$$

Thus a *fuzzy category* $L - \mathcal{F}SET(L) = (L - \mathcal{F}SET(L), \omega, \mu)$ is obtained.

Remark 3.6. If $F' : (X', E'_X) \to (Y, E_Y)$ is the restriction of $F : (X, E_X) \to (Y, E_Y)$ (see Remark 3.3 above) and $\mu(F) \geq \alpha$, then $\mu(F') \geq \alpha$. However, generally the restriction $F' : (X', E_{X'}) \to (Y', E_{Y'})$ of $F : (X, E_X) \to (Y, E_Y)$ may fail to satisfy the condition $\mu(F') \geq \alpha$.

3.3. **Some (fuzzy) subcategories of the fuzzy category $L - \mathcal{F}SET(L)$.** For a fixed α let $L - \mathcal{F}_\alpha SET(L)$ consist of all objects of $L - \mathcal{F}SET(L)$ and its fuzzy α-morphisms. In case α is idempotent, $L - \mathcal{F}_\alpha SET(L)$ is a usual (crisp) category. In particular, it is a crisp category for $\alpha = \top$.

If $L_1, L_2, L_3 \subset L$, then by $L_1 - \mathcal{F}SET(L_2, L_3)$ we denote the (fuzzy) subcategory of $L - \mathcal{F}SET(L)$, whose objects (X, E, A) satisfy the conditions $A(X) \subset L_1$ and $E(X \times X) \subset L_2$, and whose morphisms satisfy the condition $F(X \times Y) \subset L_3$. By specifying the sets L_1, L_2 and L_3 some known and new (fuzzy) categories related to L-sets can be characterized as (fuzzy) subcategories of $L_1 - \mathcal{F}SET(L_2, L_3)$-type or of $L_1 - \mathcal{F}SET'(L_2, L_3)$-type.

4. Elementary properties of fuzzy functions. Special types of fuzzy functions.

4.1. **Images and preimages of L-sets under fuzzy functions.** Assume that the GL-monoid $(L, \wedge, \vee, *)$ is equipped with an additional operation $\odot$ which is distributive over arbitrary joins and meets and is dominated by $*$, i.e. $(\alpha_1 \odot \beta_1) * (\alpha_2 \odot \beta_2) \leq (\alpha_1 * \beta_1) \odot (\alpha_2 * \beta_2)$. In particular, $\wedge$ can be taken as $\odot$. Another option: in case when $(L, \wedge, \vee, *)$ is an MV-algebra, the original conjunction $*$ can be taken as $\odot$. Given a fuzzy function $F : (X, E_X) \to (Y, E_Y)$ and L-subsets $A : X \to L$ and $B : Y \to L$ of X and Y respectively, we define the fuzzy set $F^{\to}(A) : Y \to L$ (the image of A under F) by the equality $F^{\to}(A)(y) = \bigvee_x F(x, y) \odot A(x)$ and the fuzzy set $F^{\leftarrow}(B) : X \to L$ (the preimage of B under F) by the equality $F^{\leftarrow}(B)(x) = \bigvee_y F(x, y) \odot B(y)$.

Note that if $A \in L^X$ is extensional, then $F^{\to}(A) \in L^Y$ is extensional (by (2ff)) and if $B \in L^Y$ is extensional, then $F^{\leftarrow}(B) \in L^X$ is extensional (by (3′ff)).

Proposition 4.1 (Basic properties of images and preimages of L-sets under fuzzy functions).

(1) $F^{\to}(\bigvee_{i \in \mathcal{I}}(A_i)) = \bigvee_{i \in \mathcal{I}} F^{\to}(A_i) \quad \forall \{A_i : i \in \mathcal{I}\} \subset L^X$;
(2) $F^{\to}(A_1 \bigwedge A_2) \leq F^{\to}(A_1) \bigwedge F^{\to}(A_2) \quad \forall A_1, A_2 \in L^X$;
(3) $F^{\leftarrow}(\bigwedge_{i \in \mathcal{I}}(B_i)) \leq \bigwedge_{i \in \mathcal{I}} F^{\leftarrow}(B_i) \quad \forall \{B_i : i \in \mathcal{I}\} \subset L^Y$.
(3^0) *In case L is completely distributive*

$$\Big(\bigwedge_{i \in \mathcal{I}} F^{\leftarrow}(B_i)\Big)^5 \leq F^{\leftarrow}\Big(\bigwedge_{i \in \mathcal{I}}(B_i)\Big) \leq \bigwedge_{i \in \mathcal{I}} F^{\leftarrow}(B_i) \quad \forall \{B_i : i \in \mathcal{I}\} \subset L^Y;$$

in particular,

($3^0_\wedge$) $(\bigwedge_{i\in\mathcal{I}} F^{\leftarrow}(B_i))^3 \leq F^{\leftarrow}(\bigwedge_{i\in\mathcal{I}}(B_i)) \leq \bigwedge_{i\in\mathcal{I}} F^{\leftarrow}(B_i) \quad \forall\{B_i : i \in \mathcal{I}\} \subset L^X$, *in case* $\odot = \wedge$ *and*

($3^0_{\wedge\wedge}$) $\bigwedge_{i\in\mathcal{I}} F^{\leftarrow}(B_i) = F^{\leftarrow}(\bigwedge_{i\in\mathcal{I}}(B_i)) \quad \forall\{B_i : i \in \mathcal{I}\} \subset L^Y$, *in case* $\odot = * = \wedge$;

(4) $F^{\leftarrow}(\bigvee_{i\in\mathcal{I}}(B_i)) = \bigvee_{i\in\mathcal{I}} F^{\leftarrow}(B_i) \quad \forall\{B_i : i \in \mathcal{I}\} \subset L^Y$;

(5^0_*) *In case* L *is completely distributive and* $\odot = *$, $F^{\leftarrow}(F^{\leftarrow}(B)) \leq B$.

Proof. (1):

$$\begin{aligned}\big(\bigvee_i F^{\rightarrow}(A_i)\big)(y) &= \bigvee_i \bigvee_x \big(F(x,y) \odot A_i(x)\big)\\ &= \bigvee_x \bigvee_i \big(F(x,y) \odot A_i(x)\big)\\ &= \bigvee_x (F(x,y) \odot (\bigvee_i A_i)(x))\\ &= F^{\rightarrow}(\vee_i A_i)(y).\end{aligned}$$

The validity of (2) follows from the monotonicity of F.

To prove (3) notice that

$$\begin{aligned}(\bigwedge_i F^{\leftarrow}(B_i))(x) &= \bigwedge_i \big(\bigvee_y F(x,y) \odot B_i(y)\big)\\ &\geq \bigvee_y (\bigwedge_i (F(x,y) \odot B_i(y))\\ &\geq \bigvee_y (F(x,y) \odot (\bigwedge_i B_i(y)))\\ &= F^{\leftarrow}(\bigwedge_i B_i)(x).\end{aligned}$$

Assume now that L is completely distributive. Recall that complete distributivity of a lattice L means that the way-below relation $\ll$ in L is approximative (i.e. $\alpha = \bigvee\{\beta \in L : \beta \ll \alpha\}$ for every $\alpha \in L$) and every element α is a supremum of coprimes way-below α (see e.g. [3]). Let

$$(\bigwedge_i F^{\leftarrow}(B_i))(x) = \bigwedge_i \bigvee_y F(x,y) \odot B_i(y) := \alpha.$$

Then

$$\forall \beta \ll \alpha, \forall i \in \mathcal{I}, \exists y_i \in Y \text{ such that } F(x,y_i) \odot B_i(y_i) \geq \beta.$$

In particular, this means that $F(x,y_i) \geq \beta$ for every $i \in \mathcal{I}$. We fix some $i_0 \in \mathcal{I}$ and let $y_{i_0} := y_0$. Further, notice that by Remark 3.2

$$\beta^2 \leq F(x,y_i) * F(x,y_0) \leq E(y_i,y_i) \longmapsto E(y_i,y_0),$$

and hence for every $i \in \mathcal{I}$

$$\begin{aligned}[F(x,y_i) \odot B_i(y_i)] * \beta^4 &\leq \big(F(x,y_i) * (E_Y(y_i,y_i) \longmapsto E_Y(y_i,y_0))\big)\odot\\ &\qquad \big(B_i(y_i) * (E_Y(y_i,y_i) \longmapsto E_Y(y_i,y_0))\big)\\ &\leq F(x,y_0) \odot B_i(x,y_0).\end{aligned}$$

Therefore

$$\begin{aligned}\beta^5 &\leq \bigwedge_i \big[F(x,y_i) \odot B_i(y_i)\big] * \beta^4\\ &\leq \bigwedge_i \big(F(x,y_0) \odot B_i(y_0)\big)\\ &= F(x,y_0) \odot \bigwedge_i B_i(y_0)\\ &\leq F^{\leftarrow}(\bigwedge_i B_i)(x)\end{aligned}$$

and, since this holds for any $\beta \ll \alpha$, by complete distributivity we obtain $F^{\leftarrow}(\bigwedge_i B_i)(x) \geq \alpha^5$ and hence

$$(\bigwedge_{i\in\mathcal{I}} F^{\leftarrow}(B_i))^5 \leq F^{\leftarrow}(\bigwedge_{i\in\mathcal{I}}(B_i)).$$

In case $\odot = \wedge$ in the above proof it is sufficient to multiply by β^2 instead of β^4, and therefore the resulting inequality is

$$\Big(\bigwedge_{i\in\mathcal{I}} F^{\leftarrow}(B_i)\Big)^3 \le F^{\leftarrow}\Big(\bigwedge_{i\in\mathcal{I}}(B_i)\Big).$$

Finally, in case $\odot = * = \wedge$ by idempotency we get the equality

$$\bigwedge_{i\in\mathcal{I}} F^{\leftarrow}(B_i) = F^{\leftarrow}\Big(\bigwedge_{i\in\mathcal{I}}(B_i)\Big).$$

The proof of (4) is similar to the proof of (1) and is therefore omitted.

To prove (5) assume that $F^{\leftarrow}(F^{\leftarrow}(B))(y_0) \ge \alpha$, for some $y_0 \in Y, \alpha \in L$, then for each $\beta \ll \alpha$ there exist $x_0, y_1 \in Y$ such that $F(x_0,y_0) * F(x_0,y_1) * B(y_1) \ge \beta$. Therefore, by extensionality of B:

$$\begin{aligned} B(y_0) &\ge (E(y_1,y_1) \longmapsto E(y_1,y_0)) * B(y_1)\\ &\ge F(x_0,y_0) * F(x_0,y_1) * B(y_1)\\ &\ge \beta, \end{aligned}$$

and hence, since L is completely distributive, it follows that

$$B(y_0) \ge F^{\leftarrow}(F^{\leftarrow}(B))(y_0)$$

and thus $B \ge F^{\leftarrow}(F^{\leftarrow}(B))$. □

4.2. **Injectivity, surjectivity and bijectivity of fuzzy functions.** A fuzzy function $F : (X, E_X, A) \to (Y, E_Y, B)$ is called *injective*, if

(inj) $F(x,y) * (E_Y(y,y) \longmapsto F(x',y)) \le E_X(x,x') \quad \forall x,x' \in X, \forall y \in Y.$

Notice that injective fuzzy functions satisfy the following condition

(inj#) $F(x,y) * F(x',y) \le (E_X(x,x) \vee E_X(x',x')) \longmapsto E_X(x,x') \quad \forall x,x' \in X, \forall y \in Y.$

Indeed,

$$\begin{aligned} F(x,y) * F(x',y) &\le F(x,y) * (E(y,y) \longmapsto F(x',y))\\ &\le E(x,x')\\ &\le (E(x,x) \longmapsto E(x,x')). \end{aligned}$$

Notice, that in case when E_Y is global, then (inj) just means that $F(x,y) * F(x',y) \le E_X(x,x')$.

A fuzzy function $F : (X, E_X, A) \to (Y, E_Y, B)$ is called *α-surjective* if it satisfies the following two conditions:

(sur1$^\alpha$) $\inf_y \sup_x F(x,y) \ge \alpha$

(sur2) $F^{\rightarrow}(A) \ge B \odot \alpha$.

In case F is injective and α-surjective, it is called *α-bijective.*

Remark 4.2. Notice that in case $A = \top_X$ the second condition in the definition of α-surjectivity (for any $B \in L^Y$, in particular, for $B = \top_Y$) follows from the first one. Moreover, in case $A = \top_X$, $B = \top_Y$ and if $\top$ acts as a unit with respect to $\odot$, the both conditions become equivalent.

Remark 4.3. Let $(X, E_X), (Y, E_Y)$ be L-valued sets and $(X', E_{X'})$, $(Y', E_{Y'})$ be their subspaces. Obviously, the restriction $F' : (X', E_{X'}) \to (Y', E_{Y'})$ of an injection $F : (X, E_X) \to (Y, E_Y)$ is an injection. The restriction $F' : (X, E_X) \to (Y', E_{Y'})$ of an α-surjection $F : (X, E_X) \to (Y, E_Y)$ is an α-surjection. On the other hand, generally the restriction $F' : (X', E_{X'}) \to (Y', E_{Y'})$ of an α-surjection $F : (X, E_X) \to (Y, E_Y)$ may fail to be an α-surjection.

A fuzzy function $F : (X, E_X, A) \to (Y, E_Y, B)$ defines a fuzzy *relation* $F^{-1} : (Y, E_Y, B) \to (X, E_X, A)$ by setting $F^{-1}(y, x) = F(x, y) \quad \forall x \in X, \forall y \in Y$.

Proposition 4.4 (Basic properties of injections, α-surjections and α-bijections).

(1) F^{-1} *is a fuzzy function iff* F *is injective (actually* F^{-1} *satisfies* (4ff) *iff* F *satisfies* (inj)*)*

(2) F *is* α*-bijective iff* F^{-1} *is* α*-bijective.*

(3) *if* L *is completely distributive and* F *satisfies* (inj#), *then*

$$(\bigwedge_i F^{\to}(A_i))^5 \leq F^{\to}(\bigwedge_i A_i) \leq \bigwedge_i F^{\to}(A_i) \qquad \forall \{A_i : i \in \mathcal{I}\} \subset L^X.$$

In particular,

($3_\wedge$) $(\bigwedge_i F^{\to}(A_i))^3 \leq F^{\to}(\bigwedge_i A_i) \leq \bigwedge_i F^{\to}(A_i)$ *if* $\odot = \wedge$ *and;*

($3^\wedge_\wedge$) $F^{\to}(\bigwedge_i A_i) = \bigwedge_i F^{\to}(A_i)$ *in case* $\odot = \wedge = *$;

(4) *If* F *is* $\top$*-surjective, then* $F^{\leftarrow}(F^{\leftarrow}(B)) \geq B \quad \forall B \in L^Y$; *and hence, in particular,* $F^{\leftarrow}(F^{\leftarrow}(B)) = B$ *in case* $\odot = *$ *and* L *is completely distributive.*

Proof. The validity of (1) and (2) is obvious.

To show (3) fix $y \in Y$ and let $(\wedge_i F^{\to}(A_i))(y) \geq \alpha$. Then for each coprime $\beta \ll \alpha$ and each $\in \mathcal{I}$ one can find $x_i \in X$ such that $F(x_i, y) \odot A_i(x_i) \geq \beta$ and hence, in particular, $F(x_i, y) \geq \beta$. We fix some i_0 and denote $x_{i_0} := x_0$, $A_{i_0} ;= A_0$. By (inj#) it is easy to conclude that $E_X(x_i, x_i) \longmapsto E_X(x_0, x_i) \geq \beta^2$. Now, by extensionality of all A_i we get

$$\begin{array}{rcl}
\beta^5 & \leq & \bigwedge_i \big[(F(x_i, y) \odot A_i(x_i)) * \beta^4\big] \\
& \leq & (F(x_0, y) \odot A_0(x_0)) \wedge \big[\bigwedge_{i \neq i_0} \big((F(x_i, y) * \beta^2\big) \odot (A_i(x_i) * \beta^2)\big] \\
& \leq & (F(x_0, y) \odot A_0(x_0)) \wedge \\
& & \quad \big[\bigwedge_{i \neq i_0} \quad \big(F(x_i, y) * (E(x_i, x_i) \longmapsto E(x_i, x_0))\big) \odot \\
& & \qquad \qquad \big(A_i(x_i) * (E(x_i, x_i) \longmapsto E(x_i, x_0))\big)\big] \\
& \leq & (F(x_0, y) \odot A_0(x_0)) \wedge \big[\bigwedge_{i \neq i_0} \big((F(x_0, y)\big) \odot A_i(x_0)\big] \\
& = & \bigwedge_i F(x_0, y) \odot A_i(x_0) \\
& = & F(x_0, y) \odot \bigwedge_i A_i(x_0) \\
& \leq & F\big(\wedge_i A_i\big)(y).
\end{array}$$

Since this holds for any $\beta \ll \alpha$ and L is completely distributive we get

$$(\bigwedge_i F^{\to}(A_i)))^5 \leq F^{\to}(\bigwedge_i A_i) \leq \bigwedge_i F^{\to}(A_i).$$

To show (4) let $B(y_0) \geq \alpha$. Then

$$\begin{array}{rcl} F^{\leftarrow}(F^{\leftarrow}(B))(y_0) & = & \bigvee_x \big(F(x,y_0) \odot \big(F^{\leftarrow}(B)\big)(x)\\ & = & \bigvee_x \bigvee_y \big(F(x,y_0) \odot F(x,y) \odot B(y)\big)\\ & \geq & \bigvee_x \big(F(x,y_0) \odot F(x,y_0) \odot B(y_0)\big). \end{array}$$

Now, by $\top$-surjectivity of F we complete the proof noticing that

$$F^{\leftarrow}(F^{\leftarrow}(B))(y_0) \geq \top \odot \top \odot B(y_0) \geq B(y_0).$$

□

Proposition 4.5. *Let $F : X \times Y \to L$ be a fuzzy function and $\mu(F) \geq \alpha$. Then for each coprime $\beta \ll \alpha$ there exists $Z \subset Y$ such that the restriction $G := F\mid_{X\times Z}: X \times Z \to L$ is a β-surjection and $\mu(G) \geq \beta$.*

Proof. Given coprime $\beta \ll \alpha$, let $Z := \{y \mid \exists x \in X \text{ such that } (x,y) \geq \beta\}$, and let $G := F : X \times Z \to L$ be the restriction of F to $X \times Z$.

To show that $\mu(G) \geq \beta$ assume that, contrary, $\inf_x \sup_{y\in Z} F(x,y) = \mu(G) \not\geq \beta$. Then there would exist $x_0 \in X$ such that $F(x,y) \not\geq \beta$ for each $y \in Z$. On the other hand, from $\mu(F) \geq \alpha \gg \beta$, it follows that for each $x \in X$, in particular, for x_0 there exists $y_0 \in Y$ such that $F(x_0,y_0) \geq \beta$. Besides, by definition of Z it is clear that $y_0 \in Z$. The obtained contradiction implies that $\mu(G) \geq \beta$.

To show that G is β-surjective, assume that $\inf_{y\in Z} \sup_{x\in X} G(x,y) \not\geq \beta$. It follows from here that there exists $y_0 \in Z$ such that $\sup_{x\in X} G(x,y_0) = \sup_{x\in X} F(x,y_0) \not\geq \beta$. However, this contradicts the definition of Z. Thus the first condition of the definition of β-surjectivity holds. To conclude the proof it is sufficient to apply Remark 4.2.

□

Problem 4.6. *Is it true (at least in the case of a completely distributive lattice L), that given a fuzzy function $F : X \times Y \to L$ where $\mu(F) \geq \alpha$ there exists $Z \subset Y$ such that the restriction $G := F\mid_{X\times Z}: X \times Z \to L$ is an α-surjection and $\mu(G) \geq \alpha$?*

5. CONSTRUCTIONS IN THE FUZZY CATEGORY $L - \mathcal{FSET}(L)$

5.1. **Products.** Let $L{-}\mathcal{FSET}^{\diamondsuit}(L)$ be the subcategory of $L{-}\mathcal{FSET}(L)$ having the same potential objects as $L-\mathcal{FSET}(L)$ and only such potential morphisms $F : X \times Y \to L$ from $L - \mathcal{FSET}(L)$ which satisfy the following additional condition (a certain counterpart of the axiom of strictness and the weaken form of the axiom of preservation of equalities; see e.g. [6]):

($\diamondsuit$) $F(x,y) \neq 0 \Longrightarrow E(x,x) = E(y,y)$.

Let $\mathcal{Y} = \{(Y_i, E_i, B_i) : i \in \mathcal{I}\}$ be a family of L-valued sets, $Y_0 = \{(y_i)_{i\in\mathcal{I}} \in \Pi_i Y_i \mid E_i(y_i,y_i) = E_j(y_j,y_j)\, \forall i,j \in \mathcal{I}\}$, let B_0 be the restriction of $B = \prod_{i\in\mathcal{I}} B_i$ to Y_0, and let $E(y,y') = \bigwedge_i E_i(y_i,y'_i) \forall y = (y_i), y' = (y'_i) \in Y$. Further, let $\pi_i : Y_0 \to Y_i$ be the restriction of the projection $p_i : \prod_i Y_i \to Y_i$ to Y_0. The pair (Y,E) thus defined is the product of the family $\mathcal{Y}$ in the category

$L - \mathcal{F}SET^{\diamondsuit}(L)$. Indeed let $F_i : (X, E_X, A) \to (Y_i, E_{Y_i}, B_i)$, $i \in \mathcal{I}$, be a family of fuzzy functions in $L - \mathcal{F}SET^{\diamondsuit}(L)$ and let $F := \Delta_i F_i : (X, E_X, A) \to (Y_0, E_Y, B)$, be defined by $F(x, y) = \bigwedge_i F_i(x, y_i)$. Then F is a fuzzy function. Indeed, the validity of (0ff), (1ff), (3ff) and (4ff) is easy to verify directly applying the corresponding axiom for all F_i, while the validity of (2ff) is guaranteed by the condition ($\diamondsuit$) for all $F_i, i \in \mathcal{I}$. Besides, it is clear that $F_i = \pi_i \circ F$ and that $\mu(F) = \bigwedge_i \mu(F_i)$. Thus, (Y_0, E_Y, B) is indeed the product of the family (Y_i, E_{Y_i}, B_i) in $L - \mathcal{F}SET^{\diamondsuit}(L)$. Notice, that the condition $\diamondsuit$ obviously holds for the subcategory $L - \mathcal{F}SET'(L)$ of $L - \mathcal{F}SET(L)$. Moreover, if all (Y_i, E_i) are taken from $L - \mathcal{F}SET'(L)$, then $Y_0 = \prod_i Y_i$.

5.2. **Coproducts.** Let $\mathcal{X} = \{(X_i, A_i, E_i) : i \in \mathcal{I}\}$ be a family of L-valued sets, let $X_0 = \bigcup X_i$ be the disjoint sum of sets X_i and let $A_0 \in L^X$ be defined by $A_0(x) = A_i(x)$ whenever $x \in X_i$. Further, let $q_i : X_i \to X_0$ be the inclusion map. We introduce the L-equality on X_0 by setting $E(x, x') = E_i(x, x')$ if $(x, x') \in X_i \times X_i$ for some $i \in \mathcal{I}$ and $E(x, x') = 0$ otherwise (cf [6]). Then (X_0, A_0, E) is the coproduct of $\mathcal{X}$ in $L - \mathcal{F}SET(L)$ (and hence also in $L - \mathcal{F}SET^{\diamondsuit}(L)$).

Indeed, let $F_i : (X_i, A_i, E_i) \to (Y, B, E_Y)$, $i \in \mathcal{I}$, be a family of fuzzy functions in $L - \mathcal{F}SET(L)$ and let $F := \oplus_i F_i : \oplus(X_i, A_i, E_i) \to Y, B, E_Y)$ be defined by $F(x, y) = F_i(x_i, y)$ whenever $x = x_i \in X_i$. Then the direct verification shows that F is a fuzzy function, $F_i = F \circ q_i$ and $\mu(F) = \wedge_i \mu(F_i)$.

Theorem 5.1 (Factorization of a family of α-morphisms). *Let*

$$F_i : (X, E, A) \to (Y_i, E_i, B_i)$$

be a family of fuzzy α-functions in $L - \mathcal{F}SET^{\diamondsuit}(L)$. Then for every $\beta \ll \alpha$ there exists a fuzzy β-surjective β-function $G : (X, E, A) \to (Z, E_Z, C)$ and a family of usual functions $\pi_i : (Z, C, E_Z) \to (Y_i, B_i, E_i)$ separating points such that $F_i = G \circ \pi_i$ for every $i \in \mathcal{I}$.

Proof. Indeed, let $(Y, E_Y) = \prod_{i \in \mathcal{I}} (Y_i, E_i)$ be the product in $L - \mathcal{F}SET^{\diamondsuit}(L)$ and let $F = \triangle_{i \in \mathcal{I}} F_i : X \times \prod_{i \in \mathcal{I}} Y_i \to L$. Further, given $\beta \ll \alpha$, let $Z \subset Y$ and $G : X \times Z \to L$ have the same meaning as in Proposition 4.1 and let $C := G(A)$. Thus, by Proposition 4.1 we conclude that $G : (X, A, E_X) \to (Z, C, E_Z)$ is a β-surjective fuzzy function and $\mu(G) \geq \beta$. To complete the proof it is sufficient to notice that the mappings $\pi_i : Z \to Y_i$ defined as the restrictions of projections $p_i : Y \to Y_i$ separate points of Z and that $F_i = \pi_i \circ G$. $\square$

6. FUZZY CATEGORIES RELATED TO ALGEBRA AND TOPOLOGY WITH FUZZY FUNCTIONS AS MORPHISMS.

On the basis of $L - \mathcal{F}SET(L)$ some fuzzy categories related to topology and algebra can be naturally defined. Here are three examples:

Definition 6.1 (Fuzzy category $\mathcal{F}TOP(L)$). *Let (X, E_X) be an L-valued set and let $\tau_X \subset L^X$ be the (Chang-Goguen) L-topology on X,* [2], [4], [5]*; see also* [9]. *A fuzzy function $F : (X, E_X, \tau_X) \to (Y, E_Y, \tau_Y)$ is called continuous if*

$F(V) \in \tau_X$ for all $V \in \tau_Y$. L-topological spaces and continuous fuzzy mappings between them form the fuzzy category $\mathcal{F}TOP(L)$.

Definition 6.2 (Fuzzy category $\mathcal{F}FTOP(L)$)**.** *Let (X, E_X) be an L-valued set and let $\mathcal{T}_X : L^X \to L$ be the L-fuzzy topology on X,* [16], [9]. *A fuzzy function $F : (X, E_X, \mathcal{T}_X) \to (Y, E_Y, \mathcal{T}_Y)$ is called continuous if $\mathcal{T}_X(F(V)) \geq \mathcal{T}_Y(V)$ for all $V \in L^Y$. L-fuzzy topological spaces and continuous fuzzy mappings between them form the fuzzy category $\mathcal{F}FTOP(L)$.*

Definition 6.3 (A fuzzy category $L - \mathcal{F}Gr(L)$)**.** *Let X be a group and E_X be an L-valued equality on X such that $E_X(x \cdot y, x' \cdot y') \geq E_X(x, x') * E_X(y, y')$ for all $x, x', y, y' \in X$. Further, let $G_X : X \to L$ be an (extensional) L-subgroup of X (see e.g.* [10], [13]*). A fuzzy function $F : (X, E_X, G_X) \to (Y, E_Y, G_Y)$ is called a fuzzy homomorphism if $F(x \cdot x', y \cdot y') \geq F(x, y) * F(x', y')$ for all $x, x' \in X$, $y, y' \in Y$. L-subgroups of groups endowed with L-valued equalities, and fuzzy homomorphisms between them form a fuzzy category $L - \mathcal{F}Gr(L)$.*

These and some other fuzzy categories with fuzzy functions in the role of fuzzy morphisms will be studied elsewhere.

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