MIXED ŁOJASIEWICZ EXPONENTS AND LOG CANONICAL THRESHOLDS OF IDEALS

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Abstract. We study the Łojasiewicz exponent and the log canonical threshold of ideals of $\mathcal{O}_n$ when restricted to generic subspaces of $\mathbb{C}^n$ of different dimensions. We obtain effective formulas of the resulting numbers for ideals with monomial integral closure. An inequality relating these numbers is also proven.

1. Introduction

Let us denote by $\mathcal{O}_n$ the ring of holomorphic germs $f : (\mathbb{C}^n, 0) \to \mathbb{C}$ and by $\mathbf{m}_n$ the maximal ideal of $\mathcal{O}_n$. Let us fix a germ $f \in \mathcal{O}_n$ and let us suppose that f has an isolated singularity at the origin. Then there are two well-known numbers attached to f. One of them is the *Milnor number* $\mu(f)$ of f (see [34]), which is characterized as follows:

$$\mu(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{J(f)}.$$

Here $J(f)$ denotes the Jacobian ideal of f, which is the ideal of $\mathcal{O}_n$ generated by the partial derivatives $\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}$. If $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is an analytic map germ such that $g^{-1}(0) = \{0\}$, then the *Łojasiewicz exponent of* g, denoted by $\mathcal{L}_0(g)$, is defined as the infimum of those real numbers $\alpha \in \mathbb{R}_{\geqslant 0}$ for which there exists a positive constant $C > 0$ and an open neighbourhood U of $0 \in \mathbb{C}^n$ with respect to the Euclidean topology such that

$$\|x\|^{\alpha} \leqslant C \sup_i |g_i(x)| \tag{1.1}$$

for all $x \in U$. The other invariant that we referred to at the beginning is the Łojasiewicz exponent of ∇f, where ∇f denotes the gradient map $(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$. We will also refer to this number as the *Łojasiewicz exponent of* f and we will denote it by $\mathcal{L}_0(f)$.

We remark that if I is an ideal of $\mathcal{O}_n$ of finite colength and $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is an analytic map germ whose component functions form a generating system of I, then $\mathcal{L}_0(g)$ depends only on I. We denote the resulting number by $\mathcal{L}_0(I)$. Moreover, when $n = p$, by a result of Płoski [42, p. 670] it holds that $\mathcal{L}_0(g) \leqslant \dim_{\mathbb{C}} \mathcal{O}_n/\langle g_1, \dots, g_n\rangle$ and equality holds if and only if $\operatorname{rank}(Dg)(0) \geqslant n - 1$, where Dg denotes the differential matrix of g.

The Łojasiewicz exponent $\mathcal{L}_0(I)$ admits an algebraic characterization in terms of the asymptotic Samuel function that leads to the notion of Łojasiewicz exponent of an ideal

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of finite colength of an arbitrary Noetherian local ring (see [20], [29]). It is important to remark that $\mathcal{L}_0(I)=\mathcal{L}_0(\overline{I})$, where the bar denotes integral closure.

In [49] Teissier introduced the notion of μ^*-sequence of f. This is defined as the vector $\mu^*(f)=(\mu^{(n)}(f),\dots,\mu^{(1)}(f))$, where $\mu^{(i)}(f)$ denotes the Milnor number of the restriction of f to a generic subspace of $\mathbb{C}^n$ of dimension i, for $i=1,\dots,n$. That is, if $h:(\mathbb{C}^i,0)\to(\mathbb{C}^n,0)$ is a generic linear immersion, then $\mu^{(i)}(f)=\mu(f\circ h)$, for $i=1,\dots,n$ (see also [35]).

Let $(R,\mathbf{m})$ denote a Noetherian local ring. The sequence $\mu^*(f)$ was the motivation of the development of the notion of mixed multiplicity of n ideals of finite colength $I_1,\dots,I_n$ of R by Rees [44]. This number, which is denoted by $e(I_1,\dots,I_n)$, generalizes the Samuel multiplicity of an ideal. That is, when $I_1=\cdots=I_n=I$, for some ideal I of finite colength of R, then $e(I_1,\dots,I_n)=e(I)$, where $e(I)$ denotes the Samuel multiplicity of I. Therefore, if $f\in\mathcal{O}_n$ is a function germ with an isolated singularity at the origin, in [49] Teisser proved that $\mu^{(i)}(f)=e(J(f),\dots,J(f),\mathbf{m}_n,\dots,\mathbf{m}_n)$, where $J(f)$ is repeated i times and $\mathbf{m}_n$ is repeated $n-i$ times, for all $i=1,\dots,n$ (see also [50, p. 55]). In particular $\mu^{(1)}(f)=\operatorname{ord}(f)-1$ and $\mu^{(n)}(f)=\mu(f)$.

It is natural to ask if it is possible to develop a notion analogous to mixed multiplicities $e(I_1,\dots,I_n)$ in the context of Łojasiewicz exponent. This was the motivation of the first author to introduce the Łojasiewicz exponent of a set of ideals in [4]. If $(R,\mathbf{m})$ denotes a local ring of dimension n and $I_1,\dots,I_n$ are ideals of R of finite colength, or more generally, when the Rees' mixed multiplicity $\sigma(I_1,\dots,I_n)$ is finite (see Definition 2.2), then we have a notion of Łojasiewicz exponent that is attached to the family of ideals $I_1,\dots,I_n$. Let us denote the resulting number by $\mathcal{L}_0(I_1,\dots,I_n)$. If I denotes an ideal of finite colength of R such that $I_1=\cdots=I_n=I$, then $\mathcal{L}_0(I_1,\dots,I_n)=\mathcal{L}_0(I)$. If $i\in\{1,\dots,n\}$, then we can consider the number

$$\mathcal{L}_0^{(i)}(I)=\mathcal{L}_0(I,\dots,I,\mathbf{m},\dots,\mathbf{m})$$

where I is repeated i times and $\mathbf{m}$ is repeated $n-i$ times. Let us define the vector $\mathcal{L}_0^*(I)=(\mathcal{L}_0^{(n)}(I),\dots,\mathcal{L}_0^{(1)}(I))$. Using different techniques, Hickel [20] also studied the sequence $\mathcal{L}_0^*(I)$ and showed the very interesting inequality $e(I)\leqslant\mathcal{L}_0^{(1)}(I)\cdots\mathcal{L}_0^{(n)}(I)$ (see [20, Théorème 1.1]). We also point out that, if $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ is a complex analytic function germ with an isolated singularity at the origin, then in [48] Teissier showed that, for all $i\in\{1,\dots,n\}$, there exists a non-empty Zariski open set $W^{(i)}$ in the Grassmannian manifold $G_i(\mathbb{C}^n)$ of linear subspaces of $\mathbb{C}^n$ of dimension i, such that $\mathcal{L}_0(J(f|_H))$ does not depend on H whenever $H\in W^{(i)}$ (see Remark 3.10).

Let us fix a coordinate system $x_1,\dots,x_n$ in $\mathbb{C}^n$. If I denotes a monomial ideal of $\mathcal{O}_n$, that is, if I is a proper ideal of $\mathcal{O}_n$ generated by monomials, and I has finite colength, then $\overline{I}$, $e(I)$ and $\mathcal{L}_0(I)$ are expressed in terms of some geometrical feature of the Newton polyhedron $\Gamma_+(I)$ of I. We recall that $\Gamma_+(I)$ is defined as the convex hull in $\mathbb{R}^n$ of the exponents of all the monomials belonging to I (see Section 4 for details). It is known that, if I is a monomial ideal of $\mathcal{O}_n$, then $\overline{I}$ is generated by the monomials x^k such that $k\in\Gamma_+(I)$ (see for instance [24, §1.4]), where we use the notation $x^k=x_1^{k_1}\cdots x_n^{k_n}$, for any $k\in\mathbb{Z}^n_{\geqslant 0}$. Moreover, in this case, we have that $e(I)=n!\mathrm{V}_n(\mathbb{R}^n\smallsetminus\Gamma_+(I))$, where V_n denotes n-dimensional volume, and $\mathcal{L}_0(I)$ is equal to $\min\{r\geqslant 1: re_i\in\Gamma_+(I),\text{ for all } i=1,\dots,n\}$, where $e_1, \dots, e_n$ denotes the canonical basis in $\mathbb{R}^n$ (see for instance [6, Corollary 3.6]). It is also known that the *log canonical threshold of* I, denoted by $\mathrm{lct}(I)$, which is another fundamental number associated to ideals of $\mathcal{O}_n$ (see Section 5), verifies that $\frac{1}{\mathrm{lct}(I)} = \min\{\lambda > 0 : \lambda(1, \dots, 1) \in \Gamma_+(I)\}$, by virtue of a result of Howald (see [23, Example 5]).

It is very interesting and useful to have a combinatorial description of invariants associated to ideals in terms of Newton polyhedra, at least when the ideals under consideration are generated by monomials (see also [22, 25]).

In this article we have pursued several objectives. One of them is to give a description in terms of $\Gamma_+(I)$ of the sequence $\mathcal{L}_0^*(I)$ when I is a monomial ideal of $\mathcal{O}_n$ of finite colength. We also present some inequalities relating Łojasiewicz exponents and mixed multiplicities. Given an ideal I of $\mathcal{O}_n$ of finite colength, another objective of the article is to study the relation between the sequence of log canonical thresholds of the restrictions of I to linear subspaces of different dimensions with Łojasiewicz exponents and to obtain an expression for this sequence in terms of Newton polyhedra when the ideal I is monomial.

The article is organized as follows. Section 2 consists of preliminary definitions and results. Section 3 is devoted to obtaining inequalities between Łojasiewicz exponents and quotients of mixed multiplicities. In this context, the main result is Theorem 3.7, which gives a generalization of the inequalities appearing in [20, Remarque 4.3]. In the same section we see that the numbers $\nu_I^{(i)}$ defined by Hickel in [20, p. 635] in a regular local ring coincide with the numbers $\mathcal{L}_0^{(i)}(I)$ introduced in Definition 2.7 (see Lemma 3.9).

In Section 4 we describe the sequence $\mathcal{L}_0^*(I)$ in terms of $\Gamma_+(I)$ (Theorem 4.2) when I is a monomial ideal of $\mathcal{O}_n$ of finite colength. We remark that the computation of the sequence $\mathcal{L}_0^*(I)$, for arbitrary ideal I of $\mathcal{O}_n$, is a difficult problem. In Example 4.5 we compute $\mathcal{L}_0(J(f_t))$ for the known Briançon-Speder example $f_t : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ [12]. In this section we have also included a result about the invariance of the gradient Łojasiewicz exponent $\mathcal{L}_0(\nabla f_t)$ in analytic deformations $f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with constant Milnor number (Theorem 4.6).

Let I be an ideal of $\mathcal{O}_n$ of finite colength. In Section 5 we prove that $1 \leqslant \mathrm{lct}(I)\mathcal{L}_{x_1\cdots x_n}(I)$ and equality holds when $\overline{I}$ is a monomial ideal (Theorem 5.4), where $\mathcal{L}_{x_1\cdots x_n}(I)$ denotes the Łojasiewicz exponent of $x_1 \cdots x_n$ with respect to I (see the definitions introduced in Section 2 and relation (2.2)). The proof of this equality is based on the mentioned result of Howald and the expression of $\mathcal{L}_{x_1\cdots x_n}(I)$ in terms of the Newton filtration of $\mathcal{O}_n$ induced by $\Gamma_+(I)$, when $\overline{I}$ is a monomial ideal (see Theorem 5.3).

In Section 6 we study the relation between the log canonical threshold $\mathrm{lct}^{(i)}(I)$ of the ideal I restricted to a generic linear subspace of $\mathbb{C}^n$ of dimension i with the mixed Łojasiewicz exponent $\mathcal{L}^{(i)}_{x_1\cdots x_n}(I)$, for $i = 1, \dots, n$ (Theorem 6.2). Moreover, when the ideal $\overline{I}$ is monomial we give a combinatorial expression for $\mathrm{lct}^{(i)}(I)$ in terms of $\Gamma_+(I)$ (Theorem 6.3). In this case we apply the same techniques to derive an expression in terms of $\Gamma_+(I)$ for the sequence of jumping numbers of generic i-dimensional plane sections of I, for $i = 1, \dots, n$.

2. The sequence of mixed Łojasiewicz exponents

Let $a(x)$ and $b(x)$ be two function germs $(\mathbb{C}^n, x_0) \to \mathbb{R}$, where $x_0 \in \mathbb{C}^n$. Then we write $a(x) \lesssim b(x)$ *near* x_0 to denote that there exists a positive constant $C > 0$ and an open neighbourhood U of x_0 in $\mathbb{C}^n$, with respect to the Euclidean topology, such that $a(x) \leqslant C\, b(x)$, for all $x \in U$.

Let I and J be ideals of $\mathcal{O}_n$. Let $\{f_1, \dots, f_p\}$ be a generating system of J and let $\{g_1, \dots, g_q\}$ be a generating system of I. Let us consider the maps $f = (f_1, \dots, f_p) : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ and $g = (g_1, \dots, g_q) : (\mathbb{C}^n, 0) \to (\mathbb{C}^q, 0)$. We define the *Łojasiewicz exponent of I with respect to J*, denoted by $\mathcal{L}_J(I)$, as the infimum of the set

$$\big\{\alpha \in \mathbb{R}_{\geqslant 0} : \|f(x)\|^\alpha \lesssim \|g(x)\| \text{ near } 0\big\}. \tag{2.1}$$

By convention, we set $\inf \emptyset = \infty$. So if the previous set is empty, then $\mathcal{L}_J(I) = \infty$. It is straightforward to prove that the definition of $\mathcal{L}_J(I)$ does not depend on the chosen generating sets of I and J, respectively.

Let us denote by $V(I)$ the zero set germ at 0 of I. It is known that that $\mathcal{L}_J(I)$ is finite if and only if $V(I) \subseteq V(J)$ (see [29, Section 6] or [11, p. 497]). In this case $\mathcal{L}_J(I)$ is a rational number. When the ideal J is generated only by one element $h \in \mathcal{O}_n$, then we denote $\mathcal{L}_J(I)$ by $\mathcal{L}_h(I)$. In particular, if $V(I)$ is contained in $\{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}$, then $\mathcal{L}_{x_1 \cdots x_n}(I)$ exists. The number $\mathcal{L}_{x_1 \cdots x_n}(I)$ will play a special role in Section 5.

Let us suppose that the ideal I has finite colength. When $J = \mathbf{m}_n$, then we denote the number $\mathcal{L}_J(I)$ by $\mathcal{L}_0(I)$ and we refer to $\mathcal{L}_0(I)$ as the *Łojasiewicz exponent of I*.

Let J be a proper ideal of $\mathcal{O}_n$. By virtue of the results of Lejeune and Teissier in [29, Théorème 7.2], the Łojasiewicz exponent $\mathcal{L}_J(I)$ can be expressed algebraically as

$$\mathcal{L}_J(I) = \inf\left\{\frac{r}{s} : r, s \in \mathbb{Z}_{\geqslant 1},\ J^r \subseteq \overline{I^s}\right\}. \tag{2.2}$$

This fact is one of the motivations of the definition in [4] of the notion of Łojasiewicz exponent of a set of ideals. The main tool used for this definition is the mixed multiplicity of n ideals in a local ring of dimension n.

Along this section we denote by $(R, \mathbf{m})$, or simply by R, a given Noetherian local ring of dimension $n \geqslant 1$. If $I_1, \dots, I_n$ are ideals of R of finite colength, then we denote by $e(I_1, \dots, I_n)$ the mixed multiplicity of $I_1, \dots, I_n$ defined by Teissier and Risler in [49, §2]. We also refer to [24, §17.4] or [46] for the definitions and fundamental results concerning mixed multiplicities of ideals. Here we recall briefly the definition of $e(I_1, \dots, I_n)$. Under the conditions exposed above, let us consider the function $H : \mathbb{Z}^n_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ given by

$$H(r_1, \dots, r_n) = \ell\left(\frac{R}{I_1^{r_1} \cdots I_n^{r_n}}\right), \tag{2.3}$$

for all $(r_1, \dots, r_n) \in \mathbb{Z}^n_{\geqslant 0}$, where $\ell(M)$ denotes the length of a given R-module M. Then, it is proven in [49] that there exists a polynomial $P(x_1, \dots, x_n) \in \mathbb{Q}[x_1, \dots, x_n]$ of degree n such that

$$H(r_1, \dots, r_n) = P(r_1, \dots, r_n),$$

for all sufficiently large $r_1, \dots, r_n \in \mathbb{Z}_{\geqslant 0}$. Moreover, the coefficient of the monomial $x_1 \cdots x_n$ in $P(x_1, \dots, x_n)$ is an integer. This integer is called the *mixed multiplicity* of $I_1, \dots, I_n$ and is denoted by $e(I_1, \dots, I_n)$.

We remark that if $I_1, \dots, I_n$ are all equal to a given ideal I of finite colength of R, then $e(I_1, \dots, I_n) = e(I)$, where $e(I)$ denotes the Samuel multiplicity of I. If $i \in \{0, 1, \dots, n\}$, then we denote by $e_i(I)$ the mixed multiplicity $e(I, \dots, I, \mathbf{m}, \dots, \mathbf{m})$, where I is repeated i times and the maximal ideal $\mathbf{m}$ is repeated $n - i$ times. In particular $e_n(I) = e(I)$ and $e_0(I) = e(\mathbf{m})$.

If $f \in \mathcal{O}_n$ is an analytic function germ with an isolated singularity at the origin and $J(f)$ denotes the Jacobian ideal of f, then we denote by $\mu^{(i)}(f)$ the Milnor number of the restriction of f to a generic linear subspace of dimension i passing through the origin in $\mathbb{C}^n$, for $i = 0, 1, \dots, n$. In [49] Teissier showed that $\mu^{(i)}(f) = e_i(J(f))$, for all $i = 0, 1, \dots, n$. The *μ^*-sequence of f* is defined as $\mu^*(f) = (\mu^{(n)}(f), \dots, \mu^{(1)}(f))$.

If $g_1, \dots, g_r \in R$ and they generate an ideal J of R of finite colength then we denote the multiplicity $e(J)$ also by $e(g_1, \dots, g_r)$. We will need the following known result (see for instance [24, p. 345]).

Lemma 2.1. *Let $I_1, \dots, I_n$ be ideals of R of finite colength. Let $g_1, \dots, g_n$ be elements of R such that $g_i \in I_i$, for all $i = 1, \dots, n$, and the ideal $\langle g_1, \dots, g_n \rangle$ has also finite colength. Then*

$$e(g_1, \dots, g_n) \geqslant e(I_1, \dots, I_n).$$

Definition 2.2. Let $I_1, \dots, I_n$ be ideals of R. Then we define

$$\sigma(I_1, \dots, I_n) = \max_{r \in \mathbb{Z}_{\geqslant 1}} e(I_1 + \mathbf{m}^r, \dots, I_n + \mathbf{m}^r). \tag{2.4}$$

The set of integers $\{e(I_1 + \mathbf{m}^r, \dots, I_n + \mathbf{m}^r) : r \in \mathbb{Z}_{\geqslant 0}\}$ is not bounded in general. Thus $\sigma(I_1, \dots, I_n)$ is not always finite. The finiteness of $\sigma(I_1, \dots, I_n)$ is characterized in Proposition 2.3. We remark that if I_i has finite colength, for all $i = 1, \dots, n$, then $\sigma(I_1, \dots, I_n)$ equals the usual notion of mixed multiplicity $e(I_1, \dots, I_n)$.

Let us suppose that the residue field $k = R/\mathbf{m}$ is infinite. Let $I_1, \dots, I_n$ be ideals of R and let us identify $(I_1/\mathbf{m}I_1) \oplus \cdots \oplus (I_n/\mathbf{m}I_n)$ with k^s, for some $s \geqslant 1$. We say that a given property is satisfied for a *sufficiently general element of* $I_1 \oplus \cdots \oplus I_n$, when there exist a Zariski open subset $U \subseteq k^s$ such that the said property holds for all elements $(g_1, \dots, g_n) \in I_1 \oplus \cdots \oplus I_n$ such that $(\overline{g}_1, \dots, \overline{g}_n) \in U$, under the stated identification, where $\overline{g}_i$ denotes the class of g_i in $I_i/\mathbf{m}I_i$, for all $i = 1, \dots, n$.

Proposition 2.3 ([5, p. 393])**.** *Let us suppose that the residue field $k = R/\mathbf{m}$ is infinite. Let $I_1, \dots, I_n$ be ideals of R. Then $\sigma(I_1, \dots, I_n) < \infty$ if and only if there exist elements $g_i \in I_i$, for $i = 1, \dots, n$, such that $\langle g_1, \dots, g_n \rangle$ has finite colength. In this case, we have that $\sigma(I_1, \dots, I_n) = e(g_1, \dots, g_n)$ for a sufficiently general element $(g_1, \dots, g_n) \in I_1 \oplus \cdots \oplus I_n$.*

Proposition 2.3 shows that, if $\sigma(I_1, \dots, I_n) < \infty$, then $\sigma(I_1, \dots, I_n)$ is equal to the mixed multiplicity of $I_1, \dots, I_n$ defined by Rees in [43, p. 181] (see also [45]) via the notion of general extension of a local ring. Therefore, we will refer to $\sigma(I_1, \dots, I_n)$ as the *Rees' mixed multiplicity* of $I_1, \dots, I_n$.

Lemma 2.4 ([4, p. 392])**.** *Let $J_1, \dots, J_n$ be ideals of R such that $\sigma(J_1, \dots, J_n) < \infty$. Let $I_1, \dots, I_n$ be ideals of R for which $J_i \subseteq I_i$, for all $i = 1, \dots, n$. Then $\sigma(I_1, \dots, I_n) < \infty$ and*

$$\sigma(J_1, \dots, J_n) \geqslant \sigma(I_1, \dots, I_n).$$

Under the conditions of Definition 2.2, let us denote by J a proper ideal of R. From Lemma 2.4 we obtain easily that

$$\sigma(I_1,\dots,I_n)=\max_{r\in\mathbb{Z}_{\geqslant 0}}\sigma(I_1+J^r,\dots,I_n+J^r).$$

Let us suppose that $\sigma(I_1,\dots,I_n)<\infty$. Hence, we define

$$r_J(I_1,\dots,I_n)=\min\big\{r\in\mathbb{Z}_{\geqslant 0}:\sigma(I_1,\dots,I_n)=\sigma(I_1+J^r,\dots,I_n+J^r)\big\}. \tag{2.5}$$

If I is an ideal of finite colength of R then we denote $r_J(I,\dots,I)$ by $r_J(I)$. We remark that if R is quasi-unmixed, then, by the Rees' multiplicity theorem (see for instance [24, p. 222]) we have

$$r_J(I)=\min\big\{r\in\mathbb{Z}_{\geqslant 0}:J^r\subseteq\overline{I}\big\}.$$

We will denote the integer $r_{\mathbf{m}}(I)$ by $r_0(I)$.

Definition 2.5 ([7])**.** Let $I_1,\dots,I_n$ be ideals of R such that $\sigma(I_1,\dots,I_n)<\infty$. Let J be a proper ideal of R. We define the *Łojasiewicz exponent of $I_1,\dots,I_n$ with respect to J*, denoted by $\mathcal{L}_J(I_1,\dots,I_n)$, as

$$\mathcal{L}_J(I_1,\dots,I_n)=\inf_{s\geqslant 1}\frac{r_J(I_1^s,\dots,I_n^s)}{s}. \tag{2.6}$$

In accordance with mixed multiplicities of ideals, we also refer to $\mathcal{L}_J(I_1,\dots,I_n)$ as the *mixed Łojasiewicz exponent of $I_1,\dots,I_n$ with respect to J*. When $J=\mathbf{m}$ we denote this number by $\mathcal{L}_0(I_1,\dots,I_n)$.

Remark 2.6. Let us observe that, under the conditions of Definition 2.5, if I is an ideal of finite colength of R such that $I_1=\cdots=I_n=I$, then the right hand side of (2.6) can be rewritten as

$$\inf\left\{\frac{r}{s}:r,s\in\mathbb{Z}_{\geqslant 1},\,e(I^s)=e(I^s+J^r)\right\}. \tag{2.7}$$

If we assume that R is quasi-unmixed and $r,s\in\mathbb{Z}_{\geqslant 1}$, then the condition $e(I^s)=e(I^s+J^r)$ is equivalent to saying that $J^r\subseteq\overline{I^s}$, by the Rees' multiplicity theorem. Therefore in this case, we can express (2.7) as

$$\inf\left\{\frac{r}{s}:r,s\in\mathbb{Z}_{\geqslant 1},\,J^r\subseteq\overline{I^s}\right\},$$

which coincides with the usual notion of Łojasiewicz exponent $\mathcal{L}_J(I)$ of I with respect to J (see [29, Théorème 7.2]).

We also remark that, in order to define $\mathcal{L}_J(I_1,\dots,I_n)$, the condition $\sigma(I_1,\dots,I_n)<\infty$ is required. Therefore (2.6) does not apply to giving an alternative formulation of $\mathcal{L}_J(I)$ for any pair of ideals I and J of $\mathcal{O}_n$ such that $V(I)\subseteq V(J)$ (we recall that $\mathcal{L}_J(I)$ is defined in this case as the infimum of the set given in (2.1)).

As a particular case of the previous definition we introduce the following concept.

Definition 2.7. Let I be an ideal of R of finite colength and let J be a proper ideal of R. If $i\in\{1,\dots,n\}$, then we define the *i-th relative Łojasiewicz exponent of I with respect to*

J as

$$\mathcal{L}_J^{(i)}(I) = \mathcal{L}_J(\underbrace{I, \dots, I}_{i \text{ times}}, \underbrace{\mathbf{m}, \dots, \mathbf{m}}_{n-i \text{ times}}). \tag{2.8}$$

We define the *$\mathcal{L}_J^*$-vector*, or *$\mathcal{L}_J^*$-sequence*, of I as

$$\mathcal{L}_J^*(I) = \big(\mathcal{L}_J^{(n)}(I), \dots, \mathcal{L}_J^{(1)}(I)\big).$$

If $J = \mathbf{m}$, then we denote $\mathcal{L}_J^{(i)}(I)$ by $\mathcal{L}_0^{(i)}(I)$, for all $i = 1, \dots, n$, and $\mathcal{L}_J^*(I)$ by $\mathcal{L}_0^*(I)$. We will refer to $\mathcal{L}_0^*(I)$ simply as the *sequence of relative Łojasiewicz exponents of I.*

Definition 2.8. Let $(X, 0) \subseteq (\mathbb{C}^n, 0)$ be the germ at 0 of a complex analytic variety X. Let I be an ideal of $\mathcal{O}_n$ such that $V(I) \cap X = \{0\}$. Let $g_1, \dots, g_s \in \mathcal{O}_n$ be a generating system of I and let g denote the map $(g_1, \dots, g_s) : (\mathbb{C}^n, 0) \to (\mathbb{C}^s, 0)$. Then we define the *Łojasiewicz exponent of I relative to* $(X, 0)$ as the infimum of those $\alpha > 0$ such that there exists a constant $C > 0$ and an open neighbourhood U of $0 \in \mathbb{C}^n$ with respect to the Euclidean topology such that $\|x\|^\alpha \leqslant C\|g(x)\|$, for all $x \in U \cap X$. We denote this number by $\mathcal{L}_{(X,0)}(I)$.

We will study the number $\mathcal{L}_{(X,0)}(I)$ specially when $(X, 0)$ is a linear subspace of $\mathbb{C}^n$. The following known theorem will be applied in Section 4. This shows a method to determine $\mathcal{L}_{(X,0)}(I)$ in terms of an explicit desingularization of X.

Theorem 2.9. *Let $(X, 0) \subseteq (\mathbb{C}^n, 0)$ be the germ at 0 of a complex analytic variety X. Let $\pi : M \to \mathbb{C}^n$ be a proper modification so that $\pi^*(\mathbf{m}I)_0$ is formed by normal crossing divisors whose support has the irreducible decomposition $\cup_i D_i$. If*

$$(\pi^*\mathbf{m})_0 = \sum_i s_i D_i, \quad (\pi^* I)_0 = \sum_i m_i D_i, \quad s_i, m_i \in \mathbb{Z},$$

then we have

$$\mathcal{L}_{(X,0)}(I) = \max\left\{\frac{m_i}{s_i} : D_i \cap X' \neq \emptyset\right\} \tag{2.9}$$

where X' denotes the strict transform of X by π.

For a proof of the above result we refer to [21, §6]. S. Łojasiewicz showed the inequalities that bear his name in his thesis [32]. He showed in [32] several inequalities concerning the distance functions to analytic sets. In [21], H. Hironaka showed a proof of these inequalities based on the idea of resolution of singularities. Since Łojasiewicz's setup is formulated in the real context, Hironaka gave the statement corresponding to Theorem 2.9 only in the real case, but the proof is completely parallel in the complex case. This proof enables us to determine the best exponent in Łojasiewicz's inequality and thus we obtain (2.9). The proof of Theorem 2.9 also appeared in [2, Theorem 6.4] and [3, Theorem 2.5], because of the importance of the discussion and difficulty to have an access to [21] at that time. After 2008 a republishing of [21] is available.

3. Inequalities relating Łojasiewicz exponents and mixed multiplicities

This section is motivated by the results of Hickel in [20]. In this section we expose some results showing how Łojasiewicz exponents are related with quotients of mixed multiplicities; the main result in this direction is Theorem 3.7.

Proposition 3.1. *Let $(R, \mathbf{m})$ be a quasi-unmixed Noetherian local ring of dimension n. Let $I_1, \dots, I_n, J$ be ideals of R such that $\sigma(I_1, \dots, I_n) < \infty$, $\sigma(I_1, \dots, I_{n-1}, J) < \infty$ and I_n has finite colength. Then*

$$\frac{\sigma(I_1, \dots, I_n)}{\sigma(I_1, \dots, I_{n-1}, J)} \leqslant \mathcal{L}_J(I_n).$$

Proof. Let $r, s \in \mathbb{Z}_{\geqslant 1}$. Let us suppose that $J^r \subseteq \overline{I_n^s}$. Then we obtain

$$r \cdot \sigma(I_1, \dots, I_{n-1}, J) = \sigma(I_1, \dots, I_{n-1}, J^r) \tag{3.1}$$

$$\geqslant \sigma(I_1, \dots, I_{n-1}, I_n^s) = s \cdot \sigma(I_1, \dots, I_{n-1}, I_n). \tag{3.2}$$

We refer to [4, Lemma 2.6] for equality (3.1) and to Lemma 2.1 for the inequality in (3.2). In particular

$$\frac{r}{s} \geqslant \frac{\sigma(I_1, \dots, I_{n-1}, I_n)}{\sigma(I_1, \dots, I_{n-1}, J)}.$$

By [29, Théorème 7.2] we have $\mathcal{L}_J(I_n) = \inf\{\frac{r}{s} : r, s \in \mathbb{Z}_{\geqslant 1}, J^r \subseteq \overline{I_n^s}\}$ (see Remark 2.6). Then the result follows. □

Corollary 3.2. *Let $(R, \mathbf{m})$ be a quasi-unmixed Noetherian local ring of dimension n. Let I be an ideal of finite colength of R. Then*

$$\frac{e(I)}{e_{n-1}(I)} \leqslant \mathcal{L}_0(I). \tag{3.3}$$

and equality holds if and only if

$$e_{n-1}(I)^n e(I) = e(I^{e_{n-1}(I)} + \mathbf{m}^{e(I)}).$$

Proof. Inequality (3.3) follows from applying Proposition 3.1 to the case $I_1 = \cdots = I_n = I$ and $J = \mathbf{m}$.

By the definition of $\mathcal{L}_0(I)$ we observe that equality holds in (3.3) if and only if $\mathbf{m}^{e(I)} \subseteq \overline{I^{e_{n-1}(I)}}$. This inclusion is equivalent to saying that $e(I^{e_{n-1}(I)}) = e(I^{e_{n-1}(I)} + \mathbf{m}^{e(I)})$, by the Rees' multiplicity theorem. □

Remark 3.3. Let $w = (w_1, \dots, w_n) \in \mathbb{Z}^n_{\geqslant 1}$ and let $d \in \mathbb{Z}_{\geqslant 1}$. Let us denote $\min_i w_i$ by w_0. Let $f \in \mathcal{O}_n$ denote a semi-weighted homogeneous function germ of degree d with respect to w. It is known that $\mathcal{L}_0(\nabla f) \leqslant \frac{d-w_0}{w_0}$ (see for instance [7, Corollary 4.7]). Hence it is interesting to determine when $\mathcal{L}_0(\nabla f)$ attains the maximum possible value $\frac{d-w_0}{w_0}$ (see [7, 27]).

By (3.3) we obtain

$$\frac{\mu(f)}{\mu^{(n-1)}(f)} \leqslant \mathcal{L}_0(\nabla f). \tag{3.4}$$

Therefore, if $\frac{\mu(f)}{\mu^{(n-1)}(f)} = \frac{d-w_0}{w_0}$ then we obtain the equality $\mathcal{L}_0(\nabla f) = \frac{d-w_0}{w_0}$.

Let $f_t : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ denote the analytic family of functions of Briançon-Speder's example (see Example 4.5). We recall that f_t is weighted homogeneous of degree 15 with respect to $w = (1, 2, 3)$, for all t. When $t \neq 0$, equality holds in (3.4) and thus we observe that inequality (3.3) is sharp. However $\mathcal{L}_0(\nabla f_0) = \frac{d-w_0}{w_0}$ but the equality does not hold in (3.4).

We also remark that the Briançon-Speder's example also shows that if $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is a weighted homogeneous function of degree d with respect to a given vector of weights $w = (w_1, \dots, w_n) \in \mathbb{Z}_{\geqslant 1}$, then we can not expect a formula for the whole sequence $\mu^*(f)$ in terms of w and d.

Corollary 3.4. *Let $(R, \mathbf{m})$ be a quasi-unmixed Noetherian local ring of dimension n. Let $I_1, \dots, I_n$ and $J_1, \dots, J_n$ be two families of ideals of R of finite colength. Then*

$$\frac{e(I_1, \dots, I_n)}{e(J_1, \dots, J_n)} \leqslant \mathcal{L}_{J_1}(I_1)\mathcal{L}_{J_2}(I_2) \cdots \mathcal{L}_{J_n}(I_n). \tag{3.5}$$

In particular, if R is regular and I is an ideal of R of finite colength, then

$$e(I) \leqslant \mathcal{L}_0(I)^n. \tag{3.6}$$

Proof. Relation (3.5) follows immediately as a recursive application of Proposition 3.1. Inequality (3.6) is a consequence of applying (3.5) by considering $I_1 = \cdots = I_n = I$, $J_1 = \cdots = J_n = \mathbf{m}$ and the equality $e(\mathbf{m}) = 1$. □

Lemma 3.5. *Let $(R, \mathbf{m})$ denote a Noetherian local ring of dimension n. Let $I_1, \dots, I_n$ be ideals of R such that $\sigma(I_1, \dots, I_n) < \infty$. Let $g \in I_n$ such that $\dim R/\langle g\rangle = n - 1$ and let $p : R \to R/\langle g\rangle$ denote the canonical projection. Then*

$$\sigma(I_1, \dots, I_n) \leqslant \sigma(p(I_1), \dots, p(I_{n-1})).$$

Proof. By Proposition 2.3, there exist $g_i \in I_i$, for $i = 1, \dots, n-1$, such that

$$\sigma(p(I_1), \dots, p(I_{n-1})) = \sigma(p(g_1), \dots, p(g_{n-1})).$$

The image in a quotient of R of a given ideal of R has multiplicity greater than or equal to the multiplicity of the given ideal (see for instance [24, Lemma 11.1.7] or [19, p. 146]). Therefore

$$\sigma(p(I_1), \dots, p(I_{n-1})) = e(p(g_1), \dots, p(g_{n-1})) \geqslant e(g_1, \dots, g_{n-1}, g) \geqslant \sigma(I_1, \dots, I_n)$$

where the last inequality is a consequence of Lemma 2.1. □

Proposition 3.6. *Let $(R, \mathbf{m})$ be a Noetherian local ring of dimension $n \geqslant 2$. Let J be a proper ideal of R and let $I_1, \dots, I_n$ be ideals of R such that $\sigma(I_1, \dots, I_n) < \infty$. Let g denote a sufficiently general element of I_n and let $p : R \to R/\langle g\rangle$ denote the canonical projection. Then*

$$\sigma(p(I_1), \dots, p(I_{n-1})) = \sigma(I_1, \dots, I_n) \tag{3.7}$$

$$\mathcal{L}_{p(J)}(p(I_1), \dots, p(I_{n-1})) \leqslant \mathcal{L}_J(I_1, \dots, I_n). \tag{3.8}$$

Proof. Let us suppose that $g \in I_n$ is a superficial element for $I_1, \dots, I_n$ according to [24, Definition 17.2.1]. In particular, the element g can be considered as a sufficiently general element of I_n, by [24, Proposition 17.2.2]. Therefore equality (3.7) holds, by a result of Risler and Teissier [24, Theorem 17.4.6] (see also [49, p. 306]). From (3.7) we obtain the following chain of inequalities, for any pair of integers $r, s \geqslant 1$:

$$\begin{aligned}
\sigma(I_1^s, \dots, I_n^s) &= s^n \sigma(I_1, \dots, I_n) = s^n \sigma(p(I_1), \dots, p(I_{n-1})) \\
&= s \cdot \sigma(p(I_1)^s, \dots, p(I_{n-1})^s) \geqslant s \cdot \sigma(p(I_1)^s + p(J)^r, \dots, p(I_{n-1})^s + p(J)^r) \\
&\geqslant s \cdot \sigma(I_1^s + J^r, \dots, I_{n-1}^s + J^r, I_n) = \sigma(I_1^s + J^r, \dots, I_{n-1}^s + J^r, I_n^s) \qquad (3.9)\\
&\geqslant \sigma(I_1^s + J^r, \dots, I_{n-1}^s + J^r, I_n^s + J^r),
\end{aligned}$$

where the inequality of (3.9) is a direct application of Lemma 3.5. In particular, we find that $r_{p(J)}(p(I_1)^s, \dots, p(I_{n-1})^s) \leqslant r_J(I_1^s, \dots, I_n^s)$, for all $s \geqslant 1$, and hence relation (3.8) follows. □

The next result shows an inequality that in some situations (see Corollary 3.8) is subtler than inequality (3.5). Moreover, Theorem 3.7 constitutes a generalization of the inequality proven by Hickel in [20, Théorème 1.1].

Theorem 3.7. *Let us suppose that $(R, \mathbf{m})$ is a quasi-unmixed Noetherian local ring. Let $I_1, \dots, I_n$ and $J_1, \dots, J_n$ two families of ideals of R of finite colength. Then*

$$\begin{aligned}
\frac{e(I_1, \dots, I_n)}{e(J_1, \dots, J_n)} \leqslant\ & \mathcal{L}_{J_1}(I_1, J_2 \dots, J_n) \mathcal{L}_{J_2}(I_2, I_2, J_3 \dots, J_n) \mathcal{L}_{J_3}(I_3, I_3, I_3, J_4 \dots, J_n) \\
& \cdots \mathcal{L}_{J_{n-1}}(I_{n-1}, \dots, I_{n-1}, J_n) \mathcal{L}_{J_n}(I_n, \dots, I_n).
\end{aligned}$$

Proof. By Proposition 3.1, we have

$$e(I_1, \dots, I_n) \leqslant e(I_1, \dots, I_{n-1}, J_n) \mathcal{L}_{J_n}(I_n). \qquad (3.10)$$

Let $g_n \in J_n$ such that $\dim R/\langle g_n \rangle = n-1$ and let $p : R \to R/\langle g_n \rangle$ be the natural projection. Therefore we obtain

$$e(I_1, \dots, I_{n-1}, J_n) \leqslant e(p(I_1), \dots, p(I_{n-1})), \qquad (3.11)$$

by Lemma 3.5. Applying again Proposition 3.1 we have

$$\begin{aligned}
e(p(I_1), \dots, p(I_{n-1})) &\leqslant e(p(I_1), \dots, p(I_{n-2}), p(J_{n-1})) \mathcal{L}_{p(J_{n-1})}(p(I_{n-1})) \\
&\leqslant e(p(I_1), \dots, p(I_{n-2}), p(J_{n-1})) \mathcal{L}_{J_{n-1}}(I_{n-1}, \dots, I_{n-1}, J_n), \qquad (3.12)
\end{aligned}$$

where (3.12) follows from Proposition 3.6. Thus joining (3.10), (3.11) and (3.12) we obtain

$$e(I_1, \dots, I_n) \leqslant e\left(p(I_1), \dots, p(I_{n-2}), p(J_{n-1})\right) \mathcal{L}_{J_{n-1}}(I_{n-1}, \dots, I_{n-1}, J_n) \mathcal{L}_{J_n}(I_n).$$

Now we can bound the multiplicity $e(p(I_1), \dots, p(I_{n-2}), p(J_{n-1}))$ by applying the same argument. Then, by finite induction we construct a sequence of elements $g_i \in J_i$, for $i = 2, \dots, n$, such that $\dim R/\langle g_i, \dots, g_n \rangle = i - 1$, for all $i = 2, \dots, n$, and if q denotes the projection $R \to R/\langle g_2, \dots, g_n \rangle$, then

$$\begin{aligned}
e(I_1, \dots, I_n) \leqslant\ & e(q(I_1)) \mathcal{L}_{J_2}(I_2, I_2, J_3 \dots, J_n) \mathcal{L}_{J_3}(I_3, I_3, I_3, J_4 \dots, J_n) \\
& \cdots \mathcal{L}_{J_{n-1}}(I_{n-1}, \dots, I_{n-1}, J_n) \mathcal{L}_{J_n}(I_n, \dots, I_n).
\end{aligned}$$

By Propositions 3.1 and 3.6 we have

$$e(q(I_1)) \leqslant e(q(J_1))\mathcal{L}_{q(J_1)}(q(I_1)) \leqslant e(q(J_1))\mathcal{L}_{J_1}(I_1, J_2, \dots, J_n).$$

Moreover, we can assume from the beginning that $g_n, g_{n-1}, \dots, g_2$ forms a superficial sequence for $J_n, J_{n-1}, \dots, J_2, J_1$, in the sense of [24, Definition 17.2.1]. In particular we have the equality $e(q(J_1)) = e(J_1, \dots, J_n)$, by [24, Theorem 17.4.6]. Thus the result follows. □

Corollary 3.8. *Let $(R, \mathbf{m})$ be a quasi-unmixed Noetherian local ring and let I and J be ideals of R of finite colength. Then*

$$\frac{e(I)}{e(J)} \leqslant \mathcal{L}_J(I, J, \dots, J)\mathcal{L}_J(I, I, J, \dots, J) \cdots \mathcal{L}_J(I, \dots, I).$$

Proof. It follows by considering $I_1 = \cdots = I_n = I$ and $J_1 = \cdots = J_n = J$ in the previous theorem. □

From Corollary 3.8 we conclude that if $f \in \mathcal{O}_n$ has an isolated singularity at the origin, then

$$\mu(f) \leqslant \mathcal{L}_0^{(1)}(\nabla f) \cdots \mathcal{L}_0^{(n)}(\nabla f).$$

We remark that Theorem 3.7 and Corollary 3.8 are suggested by [20, Remarque 4.3]. Moreover, let us observe that the numbers $\nu_I^{(i)}$ defined by Hickel in [20, p. 635] in a regular local ring coincide with the numbers $\mathcal{L}_0^{(i)}(I)$ introduced in Definition 2.7, as is shown in the following lemma.

Lemma 3.9. *Let (R, m) be a regular local ring of dimension n and infinite residue field k, $\mathrm{char}(k) = 0$. Let $x_1, \dots, x_n$ denote a regular parameter system of R. Let I be a proper ideal of R of finite colength and let $i \in \{1, \dots, n-1\}$. Then $\mathcal{L}_0^{(i)}(I)$ is equal to the Łojasiewicz exponent of the image of I in the quotient ring $R/\langle h_1, \dots, h_{n-i}\rangle$, where $h_1, \dots, h_{n-i}$ are linear forms chosen generically in $k[x_1, \dots, x_n]$.*

Proof. By [24, Proposition 17.2.2] and [24, Theorem 17.4.6], we can take generic linear forms $h_1, \dots, h_{n-i} \in k[x_1, \dots, x_n]$ in order to have $e(IR_H) = e_i(I)$, where R_H denotes the quotient ring $R/\langle h_1, \dots, h_{n-i}\rangle$. Let us denote by $\mathbf{m}_H$ the maximal ideal of R_H. By [20, Théorème 1.1], the number $\mathcal{L}_0(IR_H)$ does not depend on $h_1, \dots, h_{n-i}$. Let us denote the resulting number by $\nu_I^{(i)}$, as in [20]. We observe that

$$\begin{aligned}\mathcal{L}_0(IR_H) &= \inf\left\{\frac{r}{s} : \mathbf{m}_H^r \subseteq \overline{I^s R_H},\ \ r, s \in \mathbb{Z}_{\geqslant 1}\right\}\\ &= \inf\left\{\frac{r}{s} : e(I^s R_H) = e(I^s R_H + \mathbf{m}_H^r),\ \ r, s \in \mathbb{Z}_{\geqslant 1}\right\}.\end{aligned}$$

Moreover

$$\mathcal{L}_0^{(i)}(I) = \inf\left\{\frac{r}{s} : e_i(I^s) = e_i(I^s + \mathbf{m}^r),\ \ r, s \in \mathbb{Z}_{\geqslant 1}\right\}.$$

Let $r, s \geqslant 1$, then we have the following:

$$e_i(I^s) = s^i e_i(I) = s^i e(IR_H) = e(I^s R_H) \geqslant e(I^s R_H + \mathbf{m}_H^r) \geqslant e_i(I^s + \mathbf{m}^r),$$

where the last inequality follows from Lemma 3.5. In particular, if $e_i(I^s) = e_i(I^s + \mathbf{m}^r)$, then $e(I^s R_H) = e(I^s R_H + \mathbf{m}_H^r)$. This implies that $\mathcal{L}_0(IR_H) \leqslant \mathcal{L}_0^{(i)}(I)$ and consequently $\nu_I^{(i)} \leqslant \mathcal{L}_0^{(i)}(I)$.

Let us suppose that $\nu_I^{(i)} < \mathcal{L}_0^{(i)}(I)$. Let $r, s \geqslant 1$ such that $\nu_I^{(i)} < \frac{r}{s} < \mathcal{L}_0^{(i)}(I)$. Therefore $e_i(I^s) > e_i(I^s + \mathbf{m}^r)$. Let us consider generic linear forms $h_1, \dots, h_{n-i} \in k[x_1, \dots, x_n]$ such that $e_i(I^s) = e(I^s R_H)$, $e_i(I^s + \mathbf{m}^r) = e((I^s + \mathbf{m}^r)R_H)$ and $\nu_I^{(i)} = \mathcal{L}_0(IR_H)$, where $R_H = R/\langle h_1, \dots, h_{n-i}\rangle$. Since $\nu_I^{(i)} = \mathcal{L}_0(IR_H) < \frac{r}{s}$, then $e(I^s R_H) = e((I^s + \mathbf{m}^r)R_H)$ and hence $e_i(I^s) = e_i(I^s + \mathbf{m}^r)$, which is a contradiction. Therefore $\mathcal{L}_0^{(i)}(I) = \nu_I^{(i)}$. □

Remark 3.10. Let $f \in \mathcal{O}_n$ such that f has an isolated singularity at the origin. By [49, p. 308, Proposition 2.7], the image of the Jacobian ideal of f in the local ring of a hyperplane containing the origin and the Jacobian ideal of the restriction of f to such hyperplane have the same integral closure provided that the hyperplane is sufficiently general (see also [48, p. 275]). Therefore, by this observation and Lemma 3.9, we have $\mathcal{L}_0^{(i)}(J(f)) = \mathcal{L}_0(J(f|_H))$, for a sufficiently general subspace $H \subseteq \mathbb{C}^n$ of dimension i, for all $i = 1, \dots, n$.

Lemma 3.11. *Let $(R, \mathbf{m})$ be a quasi-unmixed Noetherian local ring and let I, J be ideals of R of finite colength such that $I \subseteq J$. Let us suppose that the residue field $k = R/\mathbf{m}$ is infinite. Let $i \in \{1, \dots, n-1\}$. If $e_{i+1}(I) = e_{i+1}(J)$, then $e_i(I) = e_i(J)$.*

Proof. Let $h_1, \dots, h_{n-i} \in \mathbf{m}$ sufficiently general elements of $\mathbf{m}$. Let us define $R_1 = R/\langle h_1, \dots, h_{n-i}\rangle$ and $R_2 = \langle h_1, \dots, h_{n-i-1}\rangle$. If $p : R \to R_1$ and $q : R \to R_2$ denote the natural projections, then $e_i(I) = e(p(I)R_1)$, $e_i(J) = e(p(J)R_1)$, $e_{i+1}(I) = e(q(I)R_2)$ and $e_{i+1}(J) = e(q(J)R_2)$. Since the ring R_2 is also quasi-unmixed (see for instance [24, Proposition B.44]), the condition $e_{i+1}(I) = e_{i+1}(J)$ implies that $\overline{q(I)} = \overline{q(J)}$, where the bar denotes integral closure in R_2, by the Rees' multiplicity theorem. In particular we have $\overline{p(I)} = \overline{p(J)}$, as an equality of integral closures in R_1. Thus $e(p(I)R_1) = e(p(J)R_1)$ and the result follows. □

Corollary 3.12. *Let $(R, \mathbf{m})$ be a quasi-unmixed Noetherian local ring. Let I be an ideal of R of finite colength let J be a proper ideal of R. Let us suppose that the residue field $k = R/\mathbf{m}$ is infinite. Then $\mathcal{L}_J^{(1)}(I) \leqslant \cdots \leqslant \mathcal{L}_J^{(n)}(I)$.*

Proof. Let us fix an index $i \in \{1, \dots, n-1\}$. Let us fix two integers $r, s \geqslant 1$ such that $e_{i+1}(I^s) = e_{i+1}(I^s + J^r)$. Then $e_i(I^s) = e_i(I^s + J^r)$, by Lemma 3.11. Hence the result follows from the definition of $\mathcal{L}_J^{(i)}(I)$. □

4. Mixed Łojasiewicz exponents of monomial ideals

Let I denote a monomial ideal of $\mathcal{O}_n$ of finite colength. In this section we derive an expression for the sequence $\mathcal{L}_0^*(I)$ in terms of the Newton polyhedron of I. Let us introduce first some preliminary definitions.

Let $v \in \mathbb{R}^n_{\geqslant 0}$, $v = (v_1, \dots, v_n)$. We define $v_{\min} = \min\{v_1, \dots, v_n\}$ and $A(v) = \{j : v_j = v_{\min}\}$. Given an index $i \in \{1, \dots, n\}$, we define $S^{(i)} = \{v \in \mathbb{R}^n_{>0} : |A(v)| \geqslant n+1-i\}$ and $S_0^{(i)} = \{v \in \mathbb{R}^n_{>0} : |A(v)| = n+1-i\}$. We observe that $S^{(1)} = S_0^{(1)} = \{(\lambda, \dots, \lambda) : \lambda > 0\}$, $S^{(n)} = \mathbb{R}^n_{>0}$ and $S_0^{(i)} = S^{(i)} \smallsetminus S^{(i-1)}$, for all $i = 1, \dots, n$, where we set $S^{(0)} = \emptyset$.

Let $A \subseteq \mathbb{Z}^n_{\geqslant 0}$, we define the *Newton polyhedron determined by* A, denoted by $\Gamma_+(A)$, as the convex hull in $\mathbb{R}^n$ of the set $\{k + v : k \in A,\ v \in \mathbb{R}^n_{\geqslant 0}\}$. A subset $\Gamma_+ \subseteq \mathbb{R}^n_{\geqslant 0}$ is called a *Newton polyhedron* when $\Gamma_+ = \Gamma_+(A)$, for some $A \subseteq \mathbb{Z}^n_{\geqslant 0}$.

Given a Newton polyhedron $\Gamma_+ \subseteq \mathbb{R}^n_{\geqslant 0}$ and a vector $v \in \mathbb{R}^n_{\geqslant 0}$, we define:

$$\ell(v, \Gamma_+) = \min\{\langle v, a\rangle : a \in \Gamma_+\}$$
$$\Delta(v, \Gamma_+) = \{a \in \Gamma_+ : \langle v, a\rangle = \ell(v, \Gamma_+)\}$$

where $\langle\,,\rangle$ stands for the standard scalar product in $\mathbb{R}^n$. The sets of the form $\Delta(v, \Gamma_+)$, where $v \in \mathbb{R}^n_{\geqslant 0}$, $v \neq 0$, are called *faces* of Γ_+; in this case we say that v *supports* $\Delta(v, \Gamma_+)$. If Δ is a face of Γ_+, then the *dimension* of Δ, denoted by $\dim(\Delta)$, is defined as the minimum dimension of an affine subspace containing Δ. If Δ is a face of Γ_+ of dimension $n-1$, then we say that Δ is a *facet* of Δ.

If $h \in \mathcal{O}_n$ and $h = \sum_k a_k x^k$ denotes the Taylor expansion of h around the origin, then the *support* of h is defined as the set $\mathrm{supp}(h) = \{k \in \mathbb{Z}^n_{\geqslant 0} : a_k \neq 0\}$. If $h \neq 0$, the *Newton polyhedron* of h, denoted by $\Gamma_+(h)$, is defined as $\Gamma_+(\mathrm{supp}(h))$. If $h = 0$, then we set $\Gamma_+(h) = \emptyset$. If I denotes an ideal of $\mathcal{O}_n$ and $g_1, \dots, g_r$ is a generating system of I, then the *Newton polyhedron of* I, denoted by $\Gamma_+(I)$, is defined as the convex hull of $\Gamma_+(g_1) \cup \dots \cup \Gamma_+(g_r)$. It is easy to check that the definition of $\Gamma_+(I)$ does not depend on the chosen generating system $g_1, \dots, g_r$ of I.

If $v \in \mathbb{R}^n_{\geqslant 0}$ and I denotes an ideal of $\mathcal{O}_n$, then we denote $\ell(v, \Gamma_+(I))$ simply by $\ell(v, I)$. Therefore, if $v = (1, \dots, 1) \in \mathbb{R}^n_{\geqslant 0}$, then $\ell(v, I) = \mathrm{ord}(I)$, where $\mathrm{ord}(I)$ is the *order of* I, that is, the maximum of those $r \geqslant 1$ such that $I \subseteq \mathbf{m}^r$. If $v \in \mathbb{R}^n_{>0}$ and the support of h is contained in the hyperplane of equation $\langle k, v\rangle = \ell(v, h)$, that is, when h is weighted homogeneous with respect to v, then we refer to $\ell(v, h)$ as the *degree of* h *with respect to* v and we also denote this number by $d_v(h)$.

Let us fix a Newton polyhedron $\Gamma_+ \subseteq \mathbb{R}^n_{\geqslant 0}$. We define the following equivalence relation in $\mathbb{R}^n_{\geqslant 0}$: if $v, v' \in \mathbb{R}^n_{\geqslant 0}$, then $v \sim v'$ if and only if $\Delta(v, \Gamma_+) = \Delta(v', \Gamma_+)$. The equivalence classes arising from $\sim$ form a collection of cones in $\mathbb{R}^n_{\geqslant 0}$. These cones form a subdivision of $\mathbb{R}^n_{\geqslant 0}$. We refer to this collection of cones as the *dual Newton polyhedron* of Γ_+.

For the proof of the following theorem we use several knowledge on toric modification. We refer to [18], for example, for several information on toric modification. Here we recall some of them:

- We can associate a variety X_Σ to a fan Σ, a collection of cones which is generated by several integral vectors, see [18, p. 263] for its definition.
- X_Σ is nonsingular if Σ is regular, that is, if each cone of Σ is generated by part of a basis of $\mathbb{Z}^n$ ([18, p. 266, Theorem 2.1]).
- If Σ' is a subdivision of Σ, then we have a proper modification $X_{\Sigma'} \to X_\Sigma$ ([18, p. 72, 276]).

Theorem 4.1. *If I is a monomial ideal of $\mathcal{O}_n$ of finite colength, then*

$$\mathcal{L}_0^{(i)}(I) = \max\left\{\frac{\ell(v, I)}{v_{\min}} : v \in S^{(i)}\right\},$$

for all $i = 1, \dots, n$.

Proof. Let us fix an index $i \in \{1, \dots, n\}$. Let H denote a generic i-dimensional linear subspace of $\mathbb{C}^n$. Let us consider the fan Σ_0 corresponding to the blow up at the origin, that is, the collection of cones $\mathbb{R}_{\geqslant 0}\boldsymbol{e}+\sum_{j\in J}\mathbb{R}_{\geqslant 0}e_j$, for $J\subset\{1,\dots,n\}$, where $\boldsymbol{e}=e_1+\cdots+e_n$ and $e_1,\dots,e_n$ is the canonical basis of $\mathbb{R}^n$.

Let us consider a regular subdivision Σ of the dual Newton polyhedron of $\Gamma_+(I)$, which is also a subdivision of Σ_0. Then we have a natural map from Σ to Σ_0, that is, a natural embedding of a cone in Σ to some cone in Σ_0 (see [18, p. 72]), which induces a map from Σ to Σ_0. Since Σ is a regular subdivision of the positive orthant, we have a toric modification $X_\Sigma\to X_{\mathbb{R}^n_{\geqslant 0}}=\mathbb{C}^n$. Take a vector a which is a generator of a 1-cone of Σ and denote by E_a the corresponding exceptional divisor of this toric modification. Then E_a meets H' if and only if the cone generated by a is in a cone of Σ_0 of dimension $\leqslant i$, where H' denotes the strict transform of H. Let us consider the restriction of the pullback of the inequality (1.1) to the strict transform H'. So Theorem 2.9 implies the result. $\square$

Let us fix a subset $\mathtt{L}\subseteq\{1,\dots,n\}$, $\mathtt{L}\neq\emptyset$. Then we define $\mathbb{R}^n_{\mathtt{L}}=\{x\in\mathbb{R}^n : x_i=0, \text{ for all } i\notin\mathtt{L}\}$. If $h\in\mathcal{O}_n$ and $h=\sum_k a_kx^k$ is the Taylor expansion of h around the origin, then we denote by $h_{\mathtt{L}}$ the sum of all terms a_kx^k such that $k\in\mathbb{R}^n_{\mathtt{L}}$; if no such terms exist then we set $h_{\mathtt{L}}=0$. Let $\mathcal{O}_{n,\mathtt{L}}$ denote the subring of $\mathcal{O}_n$ formed by all function germs of $\mathcal{O}_n$ that depend only on the variables x_i such that $i\in\mathtt{L}$. If I is an ideal of $\mathcal{O}_n$, then $I^{\mathtt{L}}$ denotes the ideal of $\mathcal{O}_{n,\mathtt{L}}$ generated by all $h_{\mathtt{L}}$ such that $h\in I$. In particular, if I is an ideal of $\mathcal{O}_n$ of finite colength then $I^{\{i\}}\neq 0$, for all $i=1,\dots,n$.

Corollary 4.2. *Let I be a monomial ideal of $\mathcal{O}_n$ of finite colength. Then, for all $i\in\{1,\dots,n\}$, we have*

$$\mathcal{L}_0^{(i)}(I)=\max\big\{\operatorname{ord}(I^{\mathtt{L}}):\mathtt{L}\subseteq\{1,\dots,n\},\,|\mathtt{L}|=n-i+1\big\}. \tag{4.1}$$

Proof. Let us fix an index $i\in\{1,\dots,n\}$ and let us denote the number on the right hand side of (4.1) by $m_i(I)$. If $v\in\mathbb{R}^n_{>0}$, then we denote the vector $\frac{1}{v_{\min}}v$ by w_v. If $w_v=(w_1,\dots,w_n)$, then we observe that $w_j=1$ whenever $j\in A(v)$ and $w_j>1$, otherwise.

By Theorem 4.1 we have

$$\mathcal{L}_0^{(i)}(I)=\max\left\{\ell(w_v,I):v\in S^{(i)}\right\}.$$

We remark that, since I is an ideal of finite colength, then $I^{\mathtt{L}}\neq 0$ and $\operatorname{supp}(I^{\mathtt{L}})\subseteq\operatorname{supp}(I)$, for all $\mathtt{L}\subseteq\{1,\dots,n\}$, $\mathtt{L}\neq\emptyset$.

Let us fix a vector $v=(v_1,\dots,v_n)\in S^{(i)}$. Then, from the inclusion $I^{A(v)}\subseteq I$, we deduce that $\ell(w_v,I)\leqslant\ell(w_v,I^{A(v)})=\operatorname{ord}(I^{A(v)})$. Using the definition of $S^{(i)}$, we obtain the disjoint union $S^{(i)}=S_0^{(i)}\cup S_0^{(i-1)}\cup\cdots\cup S_0^{(1)}$. Let us suppose that $i\geqslant 2$ and $v\in S_0^{(j)}$, for some $j\in\{1,\dots,i-1\}$. Then $|A(v)|\geqslant n-i+2$. Let v' be a vector obtained from v by replacing $|A(v)|-(n-i+1)$ components v_j, where $j\in A(v)$, by $v_{\min}+1$. The resulting vector v' verifies that $|A(v')|=n-i+1$, that is, $v'\in S_0^{(i)}$. Moreover we have $A(v')\subseteq A(v)$ and then $I^{A(v')}\subseteq I^{A(v)}$. Consequently $\operatorname{ord}(I^{A(v')}\geqslant\operatorname{ord}(I^{A(v)})$. This fact shows that $\max\{\operatorname{ord}(I^{A(v)}):v\in S^{(i)}\}$ is attained at the vectors $v\in S_0^{(i)}$. The case $i=1$ of this conclusion is obvious. Then we obtain the following:

$$\begin{aligned}\mathcal{L}_0^{(i)}(I)=\max\left\{\ell(w_v,I):v\in S^{(i)}\right\}&\leqslant\max\left\{\ell(w_v,I^{A(v)}):v\in S^{(i)}\right\}\\&=\max\left\{\operatorname{ord}(I^{A(v)}):v\in S^{(i)}\right\}\end{aligned}$$

$$= \max\left\{\operatorname{ord}(I^{A(v)}) : v \in S_0^{(i)}\right\}$$
$$= \max\left\{\operatorname{ord}(I^{\mathtt{L}}) : \mathtt{L} \subseteq \{1,\dots,n\},\ |\mathtt{L}| = n-i+1\right\}.$$

Hence $\mathcal{L}_0^{(i)}(I) \leqslant m_i(I)$. Let us see the converse inequality by proving that for any subset $\mathtt{L} \subseteq \{1,\dots,n\}$ such that $|\mathtt{L}| = n+1-i$, there exist some vector $v \in \mathbb{R}^n_{>0}$ such that $A(v) = \mathtt{L}$ and $\ell(w_v, I) = \operatorname{ord}(I^{\mathtt{L}})$.

Let us fix a subset $\mathtt{L} \subseteq \{1,\dots,n\}$ such that $|\mathtt{L}| = n+1-i$ and let $v = (v_1,\dots,v_n) \in \mathbb{R}^n$ such that $v_j = 1$ for all $j \in \mathtt{L}$ and $v_j > \operatorname{ord}(I^{\mathtt{L}})$, for all $j \notin \mathtt{L}$. Let us observe that, if $k \in \operatorname{supp}(I)$ and $x^k \notin I^{\mathtt{L}}$, then there exists some $j_0 \notin \mathtt{L}$ such that $k_{j_0} \geqslant 1$; in particular $\langle v, k\rangle > \operatorname{ord}(I^{\mathtt{L}})$. Let us denote the sum of the components of any vector $k \in \mathbb{Z}^n_{\geqslant 0}$ by $|k|$. Therefore we have

$$\ell(w_v, I) = \ell(v, I) = \min\left\{\min_{x^k \in I^{\mathtt{L}}} \langle v, k\rangle,\ \min_{x^k \in I \smallsetminus I^{\mathtt{L}}} \langle v, k\rangle\right\} = \min\left\{\min_{x^k \in I^{\mathtt{L}}} |k|,\ \min_{x^k \in I \smallsetminus I^{\mathtt{L}}} \langle v, k\rangle\right\}$$
$$= \min\left\{\operatorname{ord}(I^{\mathtt{L}}),\ \min_{x^k \in I \smallsetminus I^{\mathtt{L}}} \langle v, k\rangle\right\} = \operatorname{ord}(I^{L}).$$

Thus the result follows. □

Remark 4.3. If I denotes an ideal of finite colength of $\mathcal{O}_n$ then we observe that $\mathcal{L}_0^*(I) = \mathcal{L}_0^*(\overline{I})$. Therefore in Theorem 4.2 we can replace the ideal I by any ideal of $\mathcal{O}_n$ whose integral closure $\overline{I}$ is a monomial ideal. The ideals of $\mathcal{O}_n$ whose integral closure is a monomial ideal are characterized in [9, Theorem 2.11] and are called *Newton non-degenerate ideals.*

Example 4.4. Let us consider the monomial ideal of $\mathcal{O}_3$ given by $I = \langle x^a, y^b, z^c, xyz\rangle$, where $a, b, c \in \mathbb{Z}_{\geqslant 0}$ and $3 < a < b < c$. Using the formula $e(I) = 3!\mathrm{V}_n(\mathbb{R}^3_{\geqslant 0} \smallsetminus \Gamma_+(I))$ we obtain $e(I) = ab + ac + bc$. Moreover $\mathcal{L}_0^*(I) = (c, b, 3)$, by Theorem 4.2. We remark that $\mathcal{L}_0^*(I)$ does not depend on a.

Example 4.5. Let us consider the family $f_t : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ given by:

$$f_t(x, y, z) = x^{15} + z^5 + xy^7 + ty^6 z.$$

This is known as the Briançon-Speder's example (see [12]). We have that f_t has an isolated singularity at the origin, f_t is weighted homogeneous with respect to $w = (1, 2, 3)$ and $d_w(f_t) = 15$, for all t. Therefore $\mathcal{L}_0(\nabla f_t) = 14$, for all t, by [27]. It is known that $\mu^{(2)}(f_0) = 28$ and $\mu^{(2)}(f_t) = 26$, for all sufficiently small $t \neq 0$ (see [12]). Hence

$$\mu^*(f) = \begin{cases} (364, 28, 4) & \text{if } t = 0 \\ (364, 26, 4) & \text{if } t \neq 0. \end{cases}$$

It is straightforward to check that the ideal $J(f_0)$ is Newton non-degenerate, in the sense of [9, p. 57]. Thus the integral closure of $J(f_0)$ is a monomial ideal. That is

$$\overline{J(f_0)} = \overline{\langle x^{14}, y^7, xy^6, z^4\rangle}.$$

In particular, we can apply Theorem 4.2 to deduce

$$\mathcal{L}_0^*(\nabla f_0) = (14, 7, 4).$$

If $t \neq 0$, then $\Gamma_+(J(f_t)) = \Gamma_+(J)$, where J is the monomial ideal given by $J = \langle x^{14}, y^6, z^4, y^5z, xy^6\rangle$. Obviously $J \subseteq J(f_t)$. We observe that $e(J) = 336$, whereas $e(J(f_t)) = 364$. Since $e(J) \neq e(J(f_t))$ we conclude that the ideal $J(f_t)$ is not Newton non-degenerate. In particular, we can not apply Theorem 4.2 to obtain the sequence $\mathcal{L}_0^*(\nabla f_t)$.

Let us compute the number $\mathcal{L}_0^{(2)}(J(f_t))$, for $t \neq 0$. Let us fix a parameter $t \neq 0$. We remark that $\mathcal{L}_0^{(2)}(J(f_t))$ is equal to the Łojasiewicz exponent of the function $g(x, y) = f_t(x, y, ax + by)$, for generic values $a, b \in \mathbb{C}$, by Lemma 3.9 and [49, Proposition 2.7].

We recall that if I denotes an ideal of $\mathcal{O}_n$ of finite colength, then we denote by $r_0(I)$ the minimum of those $r \geqslant 1$ such that $\mathbf{m}^r \subseteq \overline{I}$. Using Singular [14] we observe that $r_0(J(g)) = 7$.

By a result of Płoski [41, Proposition 3.1], it is enough to compute the quotients $\frac{r_0(J(g)^s)}{s}$ only for those integers s such that $1 \leqslant s \leqslant r_0(J(g)^s) \leqslant e(J(g)) = 26$. Moreover, since $r_0(J(g)) - 1 < \mathcal{L}_0(J(g)) = \inf_{s\geqslant 1} \frac{r_0(J(g)^s)}{s}$, we can consider only the integers s such that $1 \leqslant s \leqslant \frac{e(J(g))}{r_0(J(g))-1} = \frac{26}{6} \simeq 4.3$, that is, such that $1 \leqslant s \leqslant 4$. Again, by applying Singular [14] we obtain

$$r_0(J(g)) = 7 \qquad r_0(J(g)^2) = 13 \qquad r_0(J(g)^3) = 20 \qquad r_0(J(g)^4) = 26.$$

Then

$$\mathcal{L}_0(J(g)) = \min\left\{\frac{r_0(J(g))}{1}, \frac{r_0(J(g)^2)}{2}, \frac{r_0(J(g)^3)}{3}, \frac{r_0(J(g)^4)}{4}\right\} = 6.5.$$

Summing up the above information we conclude

$$\mathcal{L}_0^*(\nabla f_t) = \begin{cases} (14, 7, 4) & \text{if } t = 0 \\ (14, 6.5, 4) & \text{if } t \neq 0. \end{cases}$$

To end this section we show a result about the constancy of $\mathcal{L}_0(\nabla f_t)$ in deformations of weighted homogeneous functions.

Theorem 4.6. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a weighted homogeneous function of degree d with respect to $w = (w_1, \dots, w_n)$ with an isolated singularity at the origin. Let $w_0 = \min\{w_1, \dots, w_n\}$. Let us suppose that*

$$\mathcal{L}_0(\nabla f) = \min\left\{\mu(f), \frac{d - w_0}{w_0}\right\}. \tag{4.2}$$

Let $f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an analytic deformation of f such that f_t has an isolated singularity at the origin, for all t. If $\mu(f_t)$ is constant, then $\mathcal{L}_0(\nabla f_t)$ is also constant.

Proof. Let us assume that f_t is not analytically trivial (otherwise the conclusion is immediate). We can assume that the deformation f_t is a subfamily of a versal deformation of f. By a result of Varchenko [51], the deformation f_t verifies $d_w(f_t) \geqslant d$, for all t, where $d_w(f_t)$ denotes the degree of f_t with respect to w. Then we have the following:

$$\frac{(d - w_1)\cdots(d - w_n)}{w_1 \cdots w_n} = \mu(f) = \mu(f_t) \geqslant \frac{(d_t - w_1)\cdots(d_t - w_n)}{w_1 \cdots w_n} \geqslant \frac{(d - w_1)\cdots(d - w_n)}{w_1 \cdots w_n}.$$

Therefore $d_w(f_t) = d$ and

$$\mu(f_t) = \frac{(d-w_1)\cdots(d-w_n)}{w_1\cdots w_n}$$

for all t. Consequently f_t is a semi-weighted homogeneous function, for all t, by [9, Theorem 3.3] (see also [16]). Then, by [7, Corollary 4.7], we obtain

$$\mathcal{L}_0(\nabla f_t) \leqslant \frac{d-w_0}{w_0}.$$

By the lower semi-continuity of Łojasiewicz exponents in μ-constant deformations (see [40]) we have

$$\min\left\{\mu(f), \frac{d-w_0}{w_0}\right\} = \mathcal{L}_0(\nabla f) \leqslant \mathcal{L}_0(\nabla f_t) \leqslant \min\left\{\mu(f_t), \frac{d-w_0}{w_0}\right\} = \min\left\{\mu(f), \frac{d-w_0}{w_0}\right\}.$$

Then the result follows. □

Since the order of a function can be seen as a Łojasiewicz exponent, that is $\mathrm{ord}(f) = \mathcal{L}_{\langle f\rangle}(\mathbf{m}_n)$, for all $f \in \mathbf{m}_n$, we can consider the previous result as a counterpart for Łojasiewicz exponents of gradient maps of the known results of O'Shea [38, p. 260] and Greuel [17, p. 164] about the constancy of the order of functions in deformations with constant Milnor number. After finishing this article we were informed by T. Krasiński about the preprint [13] on the computation of $\mathcal{L}_0(\nabla f)$ when f is weighted homogeneous.

5. Log canonical thresholds

This section is devoted to show a connection between the log canonical threshold of ideals and Łojasiewicz exponents. We start by giving the definition of log canonical threshold of an ideal and recalling some basic facts about this concept. We refer to the survey [37], or to [28], for more information about the log canonical threshold of ideals.

If $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is an analytic function germ, then the *log canonical threshold* of f, denoted by $\mathrm{lct}(f)$, is the supremum of those s so that $|f(x)|^{-2s}$ is locally integrable at 0, that is, integrable on some compact neighbourhood of 0. This definition is generalized for ideals as follows.

Definition 5.1. Let I be an ideal of $\mathcal{O}_n$. Let us consider a generating system $\{g_1, \dots, g_r\}$ of I. The *log canonical threshold* of I, denoted by $\mathrm{lct}(I)$, is defined as follows:

$$\mathrm{lct}(I) = \sup\{s \in \mathbb{R}_{\geqslant 0} : \left(|g_1(x)|^2 + \cdots + |g_r(x)|^2\right)^{-s} \text{ is locally integrable at } 0\}.$$

It is straightforward to see that this definition does not depend on the choice of a generating system of I. The *Arnold index* of I, denoted by $\mu(I)$, is defined as $\mu(I) = \frac{1}{\mathrm{lct}(I)}$ (see for instance [15, 37]).

One origin of the notion of log canonical threshold comes back to analysis on complex powers as generalized functions. M. Atiyah ([1]) showed a way to compute (candidate) poles of complex powers using resolution of singularities. This leads to the following well-known result (see for instance [37, Theorem 1.1]).

Theorem 5.2. *Let $\pi : M \to \mathbb{C}^n$ be a proper modification so that $(\pi^* I)_0 = \sum_i m_i D_i$ where D_i form a family of normal crossing divisors. Then*

$$\mathrm{lct}(I) = \min_i \frac{k_i + 1}{m_i}$$

where $K_M = \sum_i k_i D_i$ is the canonical divisor of M.

The proof is based on the following observation:

$$\int_{\|x\| \le \varepsilon} |x_1^{m_1} \cdots x_n^{m_n}|^{-2s} |x_1^{k_1} \cdots x_n^{k_n}|^2 \frac{dx \wedge d\bar{x}}{\sqrt{-1}^n} < \infty \iff m_i s < k_i + 1, \text{ for all } i = 1, \dots, n.$$

If $I \subseteq \mathbf{m}_n^r$, then

$$\mathrm{lct}(I) \leqslant \mathrm{lct}(\mathbf{m}_n^r) \leqslant \frac{\mathrm{lct}(\mathbf{m}_n)}{r} = \frac{n}{r}$$

by [37, Property 1.14]. As a consequence, we conclude that $\mathrm{lct}(I) \operatorname{ord}(I) \leqslant n$. Combining this with [37, Property 1.18], we have

$$\frac{1}{\operatorname{ord}(I)} \leqslant \mathrm{lct}(I) \leqslant \frac{n}{\operatorname{ord}(I)}.$$

We also recall that, due to a result of Howald [23, p. 2667] (see also [37, p. 415]), if I is a monomial ideal of $\mathcal{O}_n$, then

$$\mathrm{lct}(I) = \frac{1}{\min\{\lambda > 0 : \lambda \mathbf{e} \in \Gamma_+(I)\}}. \tag{5.1}$$

Next we introduce some preliminary definitions in order to show the main result of this section.

If $v \in \mathbb{Z}^n_{\geqslant 0}$, $v \neq 0$, then v is said to be *primitive* when the non-zero coordinates of v are mutually prime integers. Let us fix a Newton polyhedron $\Gamma_+ \subseteq \mathbb{R}^n_{\geqslant 0}$, let Γ be the union of all compact faces of Γ_+. Since the vertices of Γ_+ are contained in $\mathbb{Z}^n_{\geqslant 0}$, any facet of Γ_+ is supported by a unique primitive vector. Let us denote by $\mathcal{F}(\Gamma_+)$ the family of primitive vectors of $\mathbb{Z}^n_{\geqslant 0}$ that support some facet of Γ_+ and by $\mathcal{F}_0(\Gamma_+)$ the family of vectors $v \in \mathcal{F}(\Gamma_+)$ such that $\ell(v, \Gamma_+) \neq 0$. If Γ_+ is convenient, then it is straightforward to prove that $\mathcal{F}(\Gamma_+) = \mathcal{F}_0(\Gamma_+) \cup \{e_1, \dots, e_n\}$, where $e_1, \dots, e_n$ denotes the canonical basis of $\mathbb{R}^n$.

Let us suppose that $\Gamma_+ \neq \mathbb{R}^n_{\geqslant 0}$. Then $\mathcal{F}_0(\Gamma_+) \neq \emptyset$. Let $\mathcal{F}_0(\Gamma_+) = \{v^1, \dots v^r\}$, for some $r \geqslant 1$. Let M_Γ denote the minimum common multiple of $\{\ell(v^1, \Gamma_+), \dots, \ell(v^r, \Gamma_+)\}$. Then we define the *filtrating map associated to* Γ_+ (or, *to* Γ) as the map $\phi_\Gamma : \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ given by

$$\phi_\Gamma(k) = \min \left\{ \frac{M_\Gamma}{\ell(v^i, \Gamma_+)} \langle k, v^i \rangle : i = 1, \dots, r \right\}, \quad \text{for all } k \in \mathbb{R}^n_{\geqslant 0}.$$

If Δ is a face of Γ_+, then we denote by $C(\Delta)$ the cone formed by all semi-lines λz, $\lambda \in \mathbb{R}_{\geqslant 0}$, where z varies in Δ. We observe that $\phi_\Gamma(\mathbb{Z}^n_{\geqslant 0}) \subseteq \mathbb{Z}^n_{\geqslant 0}$, $\phi_\Gamma(k) = M_\Gamma$, for all $k \in \Gamma$, and the map ϕ_Γ is linear on each cone $C(\Delta)$, where Δ is any compact face of Γ_+.

Let us define the map $\nu_\Gamma : \mathcal{O}_n \to \mathbb{R}_{\geqslant 0} \cup \{+\infty\}$ by $\nu_\Gamma(h) = \min\{\phi_\Gamma(k) : k \in \operatorname{supp}(h)\}$, for all $h \in \mathcal{O}_n$, $h \neq 0$; we set $\nu_\Gamma(0) = +\infty$. We refer to ν_Γ as the *Newton filtration* induced by Γ_+ (see also [8, 9, 26] for the case where Γ_+ is convenient). If J is an ideal of $\mathcal{O}_n$, then we define $\nu_\Gamma(J) = \min\{\nu_\Gamma(g) : g \in J\}$.

Proposition 5.3. *Let I be a monomial ideal of $\mathcal{O}_n$. Let $\Gamma_+ = \Gamma_+(I)$ and let $M = M_\Gamma$. Then*

$$\mathcal{L}_J(I) = \frac{M}{\nu_\Gamma(J)},$$

for any ideal J of $\mathcal{O}_n$ such that $V(I) \subseteq V(J)$.

Proof. By [29, Théorème 7.2] we know that

$$\mathcal{L}_J(I) = \inf\left\{\frac{p}{q} : p, q \in \mathbb{Z}_{\geqslant 1},\ J^p \subseteq \overline{I^q}\right\}.$$

Let p, $q \in \mathbb{Z}_{\geqslant 1}$. Since I is a monomial ideal, then $\overline{I^q}$ is also. Therefore $J^p \subseteq \overline{I^q}$ if and only if $\Gamma_+(J^p) \subseteq \Gamma_+(\overline{I^q})$. Let us observe that $\Gamma_+(\overline{I^q}) = \Gamma_+(I^q) = q\Gamma_+(I)$. Then $J^p \subseteq \overline{I^q}$ if and only if $\nu_\Gamma(J^p) \geqslant qM$, which in turn is equivalent to saying that $\frac{p}{q} \geqslant \frac{M}{\nu_\Gamma(J)}$, since $\nu_\Gamma(J^p) = p\nu_\Gamma(J)$, and then the result follows. □

Theorem 5.4. *Let I and J be proper ideals of $\mathcal{O}_n$ such that $V(I) \subseteq V(J)$. Then*

$$\operatorname{lct}(J) \leqslant \mathcal{L}_J(I)\operatorname{lct}(I). \tag{5.2}$$

Equality holds in (5.2) *when $\overline{I}$ is a monomial ideal and $J = \langle x_1 \cdots x_n\rangle$. That is, if $\overline{I}$ is a monomial ideal then* $\operatorname{lct}(I)\mathcal{L}_{x_1\cdots x_n}(I) = 1$.

Proof. Let $\{f_1, \dots, f_p\}$ be a generating system of J and let $\{g_1, \dots, g_q\}$ be a generating system of I. Let us consider the maps $f = (f_1, \dots, f_p) : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ and $g = (g_1, \dots, g_q) : (\mathbb{C}^n, 0) \to (\mathbb{C}^q, 0)$. If $\|f(x)\|^\theta \lesssim \|g(x)\|$, for some $\theta \geqslant 0$ and we fix any $s \geqslant 0$ then

$$\int_K \|g(x)\|^{-2s}\frac{dx \wedge d\bar{x}}{\sqrt{-1}^n} \lesssim \int_K \|f(x)\|^{-2s\theta}\frac{dx \wedge d\bar{x}}{\sqrt{-1}^n},$$

where K denotes some compact neighbourhood of 0 in $\mathbb{C}^n$. This shows that if $s\theta \leqslant \operatorname{lct}(J)$ then $s \leqslant \operatorname{lct}(I)$, that is, $\operatorname{lct}(J)/\theta \leqslant \operatorname{lct}(I)$. We thus obtain the inequality $\operatorname{lct}(J) \leqslant \theta \operatorname{lct}(I)$ and (5.2) follows.

Let us suppose that $\overline{I}$ is a monomial ideal. Let $\Gamma_+ = \Gamma_+(I)$ and let $M = M_\Gamma$. Let us recall that in this case $\overline{I}$ is equal to the ideal generated by all the monomials x^k such that $k \in \Gamma_+$ (see [24, p. 11]). It follows easily from the definition of log canonical threshold and Łojasiewicz exponent that $\operatorname{lct}(I) = \operatorname{lct}(\overline{I})$ and $\mathcal{L}_{x_1\cdots x_n}(I) = \mathcal{L}_{x_1\cdots x_n}(\overline{I})$. Then we can suppose that I is an integrally closed monomial ideal.

Let $\mathbf{e}$ denote the vector $(1, \dots, 1) \in \mathbb{R}^n$. Let $\lambda_0 = \min\{\lambda \in \mathbb{R}_{>0} : \lambda\mathbf{e} \in \Gamma_+\}$. We observe that, if $\lambda \in \mathbb{R}_{>0}$, then $\lambda\mathbf{e} \in \Gamma_+$ if and only if $\phi_\Gamma(\lambda\mathbf{e}) \geqslant M$. Then $\lambda_0 = M/\phi_\Gamma(\mathbf{e})$. Hence we obtain, by Lemma 5.3, that

$$\mathcal{L}_{x_1\cdots x_n}(I) = \frac{M}{\nu_\Gamma(x_1 \cdots x_n)} = \frac{M}{\phi_\Gamma(\mathbf{e})} = \lambda_0 = \frac{1}{\operatorname{lct}(I)} \tag{5.3}$$

where the last equality is an application of (5.1). □

Example 5.5. Let us consider the ideal $I = \langle x + y, xy\rangle$ of $\mathbb{C}[[x, y]]$. Then $\mathcal{L}_{xy}(I) = 1$ and $\operatorname{lct}(I) = 3/2$. We remark that $\overline{I} = \langle x + y\rangle + \langle x, y\rangle^2$. Hence, taking $J = \langle xy\rangle$, this example shows that, in general, equality does not hold in (5.2).

Let us remark that Theorem 5.4 does not assume that the ideal I has finite colength. If the ideal I has finite colength then, by [15, Theorem 0.1], we obtain the inequality $\mathrm{lct}(I) \geqslant \frac{n}{e(I)^{1/n}}$. Moreover, by Corollary 3.4, we know that $e(I) \leqslant \mathcal{L}_0(I)^n$. Joining both inequalities we deduce that $\mathrm{lct}(I) \geqslant \frac{n}{\mathcal{L}_0(I)}$, which also follows as an application of (5.2) for the case $J = \mathbf{m}_n$. Let us remark that, by applying (5.2) to the principal ideal $J = \langle x_1 \cdots x_n \rangle$, we obtain $\mathrm{lct}(I) \geqslant \frac{1}{\mathcal{L}_{x_1\cdots x_n}(I)}$. As a direct application of the definition of Łojasiewicz exponent we have

$$\mathcal{L}_{x_1\cdots x_n}(I) \leqslant \mathcal{L}_{\mathbf{m}_n^n}(I) = \frac{\mathcal{L}_0(I)}{n}.$$

Then we deduce again that

$$\mathrm{lct}(I) \geqslant \frac{1}{\mathcal{L}_{x_1\cdots x_n}(I)} \geqslant \frac{n}{\mathcal{L}_0(I)}.$$

We observe that, as a consequence of Theorem 5.4 and [15, Theorem 0.1], if I is an ideal of $\mathcal{O}_n$ of finite colength such that $\overline{I}$ is generated by monomials, then

$$\frac{1}{\mathcal{L}_{x_1\cdots x_n}(I)} \geqslant \frac{n}{e(I)^{1/n}}. \tag{5.4}$$

and equality holds if and only if $\overline{I} = \mathbf{m}_n^{\mathrm{ord}(I)}$. The above inequality does not hold in general. If I denotes the non-monomial ideal of Example 5.5, then the opposite inequality of (5.4) holds.

Remark 5.6. It is natural to ask when equality holds in (5.2) in general. Let us suppose that I and J are monomial ideals of $\mathcal{O}_n$ such that $V(I) \subseteq V(J)$. Let us observe that

$$\mathrm{lct}(I) = \min_{\substack{a\in\mathbb{R}^n_{\geqslant 0}\\ \ell(a,I)\neq 0}} \frac{\sum_i a_i}{\ell(a,I)}, \quad \mathrm{lct}(J) = \min_{\substack{a\in\mathbb{R}^n_{\geqslant 0}\\ \ell(a,J)\neq 0}} \frac{\sum_i a_i}{\ell(a,J)}, \quad \mathcal{L}_J(I) = \max_{\substack{a\in\mathbb{R}^n_{\geqslant 0}\\ \ell(a,I)\neq 0}} \frac{\ell(a,I)}{\ell(a,J)} \tag{5.5}$$

where the first and the second equalities follow immediately as an application of (5.1). The proof of the third equality of (5.5) is analogous to the proof of Lemma 5.3. Moreover we observe that the condition $V(I) \subseteq V(J)$ implies that $\mathcal{L}_J(I)$ exists (see [29, Section 6]), so $J^p \subseteq \overline{I^q}$, for some $p, q \in \mathbb{Z}_{\geqslant 1}$ and thus, if $a \in \mathbb{R}^n_{\geqslant 0}$ verifies that $\ell(a,J) = 0$ then $\ell(a,I) = 0$ also. Hence we deduce that, if a given vector $a \in \mathbb{R}^n_{\geqslant 0}$ attains the three equalities of (5.5), then we have $\mathrm{lct}(I) = \mathcal{L}_I(J)\,\mathrm{lct}(J)$.

It is worth recalling here part of a result of Loeser (see [30, 31, 47]), which gives also a relation between log canonical thresholds and Łojasiewicz exponents. Using our notation and Remark 3.10, if $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is an analytic function germ with an isolated singularity at the origin, then Loeser proved in [31] that

$$\sum_{i=1}^{n} \frac{1}{1 + \lceil \mathcal{L}_0^{(i)}(J(f)) \rceil} \leqslant \mathrm{lct}(f)$$

where $\lceil \alpha \rceil$ denotes the least integer greater than or equal to α, for any $\alpha \in \mathbb{R}$.

6. **Log canonical thresholds of generic sections**

Definition 6.1. Let I be an ideal of $\mathcal{O}_n$. For any integer $k \in \{0, 1, \dots, n-1\}$ we set

$$\mathrm{lct}^{(n-k)}(I) = \mathrm{lct}(I|_L),$$

where L denotes a generic $(n-k)$-dimensional linear subspace of $\mathbb{C}^n$, and $I|_L$ denotes the restriction of the ideal I to L.

By the semicontinuity of the log canonical threshold ([28, Corollary 9.5.39]), for every family $\{L_t\}_{t\in U}$ of linear subspaces of dimension $n-k$ with $L_0 = L$ there is an open neighborhood W of 0 such that $\mathrm{lct}(I|_{L_t}) \geqslant \mathrm{lct}(I|_{L_0})$ for every $t \in W$. So $\mathrm{lct}^{(n-k)}(I)$ is well-defined and is characterized as the maximal possible value of $\mathrm{lct}(I|_L)$, where L denotes a generic $(n-k)$-dimensional linear subspace of $\mathbb{C}^n$.

When L is the zero set of the linear forms $h_1, \dots, h_k$, then $\mathrm{lct}^{(n-k)}(I)$ is the log canonical threshold of the ideal generated by the image of I in $\mathcal{O}_n/\langle h_1, \dots, h_k\rangle$. By Proposition 4.5 of [36] (or Property 1.17 of [37]), we have

$$\mathrm{lct}^{(1)}(I) \leqslant \mathrm{lct}^{(2)}(I) \leqslant \cdots \leqslant \mathrm{lct}^{(n)}(I). \tag{6.1}$$

Theorem 5.4 has the following analogy for $\mathrm{lct}^{(k)}(I)$.

Theorem 6.2. *Let I be an ideal of $\mathcal{O}_n$ of finite colength. Then*

$$1 - \frac{k}{n} \leqslant \mathrm{lct}^{(n-k)}(I)\mathcal{L}^{(n-k)}_{x_1\cdots x_n}(I)$$

for all $k = 0, 1, \dots, n-1$.

Proof. Let L be a linear $(n-k)$-dimensional subspace of $\mathbb{C}^n$. Assume that I is generated by $f_1, \dots, f_m$ and set $f = (f_1, \dots, f_m)$. Let $H_i = \{h_i = 0\}$ denote a generic hyperplane of $\mathbb{C}^n$ through 0 so that $L = H_1 \cap \cdots \cap H_k$. Let ω denote an $(n-k)$-form with $dx_1 \wedge \cdots \wedge dx_n = dh_1 \wedge \cdots \wedge dh_k \wedge \omega$. Let $\pi : M \to \mathbb{C}^n$ denote the blow up at the origin and let h'_i denote the strict transform of h_i. Set $x_1 = u_1$ and $x_i = u_1u_i$ $(i = 2, \dots, n)$. Since $h_i = u_1h'_i$, then

$$dh_i = d(u_1h'_i) = u_1dh'_i + h'_idu_1 = u_1dh'_i$$

on the set defined by $h'_i = 0$. Let ω' denote an $(n-k)$-form with $du_1 \wedge \cdots \wedge du_n = dh' \wedge \omega'$. Since L is generic, the strict transform L' of L and the zeros of u_i $(i = 2, \dots, n)$ form a normal crossing variety. Since

$$\begin{aligned}(u_1dh'_1) \wedge \cdots \wedge (u_1dh'_k) \wedge \omega =& dh_1 \wedge \cdots dh_k \wedge \omega \\ =& dx_1 \wedge \cdots \wedge dx_n \\ =& u_1^{n-1}du_1 \wedge \cdots \wedge du_n \qquad \text{on } L',\end{aligned}$$

we may assume that $\omega = u_1^{n-k-1}\omega'$ on L'. If $|x_1 \cdots x_n|^\theta \lesssim \|f\|$ on L, we have

$$\begin{aligned}\int_{K\cap L} \|f\|^{-2s}\frac{\omega \wedge \bar{\omega}}{\sqrt{-1}^{n-k}} \lesssim& \int_{K\cap L} |x_1 \cdots x_n|^{-2\theta s}\frac{\omega \wedge \bar{\omega}}{\sqrt{-1}^{n-k}} \\ =& \int_{\pi^{-1}(K)\cap L'} |u_1^n u_2 \cdots u_n|^{-2\theta s}|u_1|^{2(n-k-1)}\frac{\omega' \wedge \bar{\omega}'}{\sqrt{-1}^{n-k}}\end{aligned}$$

$$= \int_{\pi^{-1}(K)\cap L'} |u_1|^{-2(n\theta s - n + k + 1)} |u_2 \cdots u_n|^{-2\theta s} \frac{\omega' \wedge \bar{\omega}'}{\sqrt{-1}^{n-k}}$$

which is integrable whenever $n\theta s < n - k$. So we have that $s < (1 - \frac{k}{n})/\mathcal{L}^{(n-k)}_{x_1 \cdots x_n}(I)$ implies $s < \mathrm{lct}^{(n-k)}(I)$, and we have

$$1 - \frac{k}{n} \leqslant \mathrm{lct}^{(n-k)}(I)\mathcal{L}^{(n-k)}_{x_1 \cdots x_n}(I).$$

□

We end the article by showing a closed formula for $\mathrm{lct}^{(k)}(I)$ when $\overline{I}$ is generated by monomials. Let us recall that, given an index $i \in \{1, \dots, n\}$, in Section 4 we defined $S^{(i)} = \{v \in \mathbb{R}^n_{>0} : |A(v)| \geqslant n + 1 - i\}$, where $A(v) = \{j : v_j = v_{\min}\}$, for all $v \in \mathbb{R}^n_{>0}$.

Theorem 6.3. *Let I be an ideal of $\mathcal{O}_n$ such that $\overline{I}$ is a monomial ideal. Then*

$$\begin{aligned} \mathrm{lct}^{(k)}(I) &= \min\Big\{ \frac{\sum_i a_i - (n-k)a_{\min}}{\ell(a, I)} : a \in S^{(k)} \Big\} \\ &= \inf\Big\{ \frac{\sum_i a_i - (n-k)}{\ell(a, I)} : a \in S^{(k)} \cap A \Big\} \end{aligned}$$

where $A = \{a = (a_1, \dots, a_n) : \min\{a_1, \dots, a_n\} = 1\}$, for all $k \in \{1, \dots, n\}$.

Proof. We may assume that I is a monomial ideal. We consider a toric modification $\sigma : X \to \mathbb{C}^n$ which dominate the blowing up at the origin. There is a coordinate system $(y_1, \dots, y_n)$ so that σ is expressed by

$$x_i = y_1^{a_i^1} \cdots y_n^{a_i^n} \qquad (a_i^j \in \mathbb{Z},\ i = 1, \dots, n).$$

Then we have $h_i = y_1^{a^1_{\min}} \cdots y_n^{a^n_{\min}} \tilde{h}_i$ where $\tilde{h}_i$ denotes the strict transform of h_i by σ. So we have

$$dh_i = y_1^{a^1_{\min}} \cdots y_n^{a^n_{\min}} d\tilde{h}_i$$

on the set defined by $\tilde{h}_i = 0$. Since

$$\begin{aligned} \Big(\wedge_{i=1}^k (y_1^{a^1_{\min}} \cdots y_n^{a^n_{\min}} d\tilde{h}_i) \Big) \wedge \omega =& dh_1 \wedge \cdots \wedge dh_k \wedge \omega \\ =& dx_1 \wedge \cdots \wedge dx_n \\ =& y_1^{\sum_i a_i^1 - 1} \cdots y_n^{\sum_i a_i^n - 1} dy_1 \wedge \cdots \wedge dy_n \end{aligned}$$

we obtain that

$$\omega = y_1^{\sum_i a_i^1 - k a^1_{\min} - 1} \cdots y_n^{\sum_i a_i^n - k a^n_{\min} - 1} \tilde{\omega}$$

where $\tilde{\omega}$ is a holomorphic $(n-k)$-form which does not vanish on the strict transform $\tilde{L}$ of L by σ with

$$dy_1 \wedge \cdots \wedge dy_n = d\tilde{h}_1 \wedge \cdots \wedge d\tilde{h}_k \wedge \tilde{\omega}.$$

Since L is generic, $\tilde{L}$ and the zeros of y_j form a normal crossing variety and we conclude that

$$\mathrm{lct}^{(n-k)}(I) = \min\Big\{ \frac{\sum_i a_i - k a_{\min}}{\ell(a, I)} : a \in S^{(n-k)} \Big\}.$$

We complete the proof by replacing k by $n - k$ in the above relation. □

We close the paper by showing a similar formula for the jumping numbers of ideals of $\mathcal{O}_n$ with monomial integral closure. If I is an ideal of $\mathcal{O}_n$ and $c \in \mathbb{Q}_{\geqslant 0}$, then we denote by $\mathcal{J}(I^c)$ the *multiplier ideal of I with exponent c*. Let us recall that $\{\mathcal{J}(I^c)\}_{c\in\mathbb{Q}_{\geqslant 0}}$ is a decreasing sequence of integrally closed ideals associated to I. There is an extensive literature concerning the sequence of multiplier ideals. We refer to [10], [33] or [37] for the definition and properties of the family $\{\mathcal{J}(I^c)\}_{c\in\mathbb{Q}_{\geqslant 0}}$. It is known (see for instance [10, Lemma 4.6]) that there exists an increasing sequence of rational numbers $0 = \xi_0 < \xi_1 < \xi_2 < \cdots$ such that $\mathcal{J}(I^c)$ is constant for $\xi_i \leqslant c < \xi_{i+1}$ and $\mathcal{J}(I^{\xi_i}) \supsetneq \mathcal{J}(I^{\xi_{i+1}})$, for all $i \geqslant 0$. The numbers ξ_i are called the *jumping numbers of I* or *jumping coefficients of I*. Let us remark that $\xi_1 = \mathrm{lct}(I)$.

It is known ([10, Example 4.7], [23]) that if I is a monomial ideal and $\partial\Gamma_+(I)$ denotes the boundary, in the usual Euclidian sense, of the Newton polyhedron $\Gamma_+(I)$ then the jumping numbers of I form the set

$$\left\{c \in \mathbb{Q}_{\geqslant 0} : \nu + \mathbf{e} \in c \cdot \partial\Gamma_+(I), \text{ for some } \nu \in \mathbb{Z}^n_{\geqslant 0}\right\}$$

where $\mathbf{1} = (1,\dots,1) \in \mathbb{R}^n$.

Let I be an ideal of $\mathcal{O}_n$ such that $\overline{I}$ is a monomial ideal. Let us consider the problem of determining the jumping numbers of generic k-dimensional plane sections of I, for $k = 1,\dots,n$. Following the same argument as in Theorem 6.3, given an element $h \in \mathcal{O}_n$ we have the following characterization:

$$h|_L \in \mathcal{J}((I|_L)^c) \Longleftrightarrow \langle a, \nu+\mathbf{e}\rangle - (n-k)\cdot a_{\min} \geqslant c\,\ell(a,I), \text{ for all } a \in S^{(k)} \text{ and all } \nu \in \Gamma_+(h).$$

Let us define, for any $\nu \in \mathbb{Z}^n_{\geqslant 0}$ and $k \in \{1,\dots,n\}$, the number $\xi^{(k)}_\nu$ by

$$\xi^{(k)}_\nu = \min\left\{\frac{\langle a, \nu+\mathbf{e}\rangle - (n-k)\cdot a_{\min}}{\ell(a,I)} : a \in S^{(k)}\right\}.$$

Therefore, as in Theorem 6.3, we conclude that, if L denotes a generic k-dimensional linear subspace of $\mathbb{C}^n$, then the jumping numbers of $I|_L$ are given by $\{\xi^{(k)}_\nu : \nu \in \mathbb{Z}^n_{\geqslant 0}\}$, $k = 1,\dots,n$.

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