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Baire property in product spaces

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Abstract

We show that if a product space Π has countable cellularity, then a dense subspace X of Π is Baire provided that all projections of X to countable subproducts of Π are Baire. It follows that if X_i is a dense Baire subspace of a product of spaces having countable π-weight, for each $i \in I$, then the product space $\prod_{i\in I} X_i$ is Baire. It is also shown that the product of precompact Baire paratopological groups is again a precompact Baire paratopological group. Finally, we focus attention on the so-called strongly Baire *spaces and prove that some Baire spaces are in fact strongly Baire.*

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1. Introduction

Baire spaces, i.e. topological spaces which satisfy the conclusion of the Baire category theorem, constitute an important class in several branches of Mathematics. All Čech-complete spaces, the pseudocompact spaces or, more generally, the regular feebly compact spaces are Baire. In a sense, the Baire property is one of the weakest forms of topological completeness.

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The class of Baire spaces is closed under taking open subspaces as well as G_δ-dense subspaces. Also, images of Baire spaces under open continuous mappings and, more generally, under d-open mappings (see [23]) are Baire. The problem of whether a product of a family of Baire spaces is Baire is an old one and is also well known that the answer to the problem is negative even for the product of two metric Baire spaces [9, 11] or two linear normed spaces [16, 26]. However, there are several cases when products (finite, countable or arbitrary) of Baire spaces are again Baire. In the following list we collect some results of this kind.

Let X and Y be Hausdorff spaces with the Baire property. Then $X \times Y$ is a Baire space provided that the second factor satisfies one of the following conditions (all the terms will be explained in Section 4):

a) Y is quasi-regular and has a countable π-base (Oxtoby [18] and Frolík [12]);
b) Y is *pseudo-complete* or *countably complete* (Aarts and Lutzer [1], Frolík [12]);
c) Y is α*-favorable* in the *Choquet game* $\mathcal{G}(Y)$ (White [29]);
d) Y is metric and *hereditarily Baire* (Moors [17]).

Examples of spaces as in a) are Baire separable metrizable spaces, as in b) are feebly compact spaces, as in c) are Čech-complete or countably complete spaces and as in d) are completely metrizable spaces. Our aim is to present some other productive classes of Baire spaces. The first of them is the class of dense Baire subspaces of products of spaces having countable π-weight. The corresponding fact is established in Corollary 2.10 which extends a theorem of Oxtoby [18] and also implies a result of Tkachuk [25, Theorem 4.8] on the productivity of the Baire property in the spaces of the form $C_p(X)$. A second productive class consists of the precompact paratopological groups with the Baire property (see Theorem 3.3). It is worth noting that the productivity of the class of precompact topological groups with the Baire property was proved earlier in [7].

In Section 4 we establish that under additional conditions, Baire spaces turn out to be *strongly Baire.* Along with the main theorem of [15] our results imply that every regular Baire semitopological group G is a topological group provided that G is either a Lindelöf Σ-space or a $w\Delta$-space (see Corollaries 4.4 and 4.6).

2. Products with the Baire property

There exist several countably productive classes of spaces that fail to be productive. This is the case of (complete) metric spaces or quasi-regular spaces with a countable π-base. It is also well known, however, that an arbitrary product of complete metric spaces has the Baire property (Bourbaki [5]), as does the product of quasi-regular Baire spaces of countable π-weight (Oxtoby [18]). Later on, Chaber and Pol [8] proved that a similar result holds for products of hereditarily Baire metric spaces. In Corollary 2.10 below we extend the result of Oxtoby to dense Baire subspaces of products of spaces with countable

π-weight. In fact, we prove a more general fact in Theorem 2.9 for spaces with a 'good' lattice of skeletal mappings and then apply it to dense subspaces of product spaces.

Let us recall that a continuous mapping $f\colon X \to Y$ is said to be *skeletal* if the preimage $f^{-1}(A)$ of every nowhere dense subset A of Y is nowhere dense in X. It is clear that f is skeletal iff $f^{-1}(U)$ is dense in X for every dense open subset U of Y iff the closure of $f(V)$ has a non-empty interior in Y for every non-empty open set V in X.

Skeletal mappings need not be surjective (consider the identity embedding of the rational numbers $\mathbb{Q}$ into the real line $\mathbb{R}$, where both spaces carry the usual interval topologies). The following fact clarifies how close a skeletal mapping has to be to surjective ones (see [27, Lemma 2.2]).

Needless to say, the mappings in the next two lemmas are not assumed to be surjective.

Lemma 2.1. *If $f\colon X \to Y$ is a skeletal mapping, then the set $\mathrm{int}_Y\, \overline{f(V)}$ is dense in $\overline{f(V)}$, for each open set V in X. In particular, $\mathrm{int}_Y\, \overline{f(X)}$ is dense in $\overline{f(X)}$.*

Lemma 2.2. *Suppose that $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous mappings. If both f and $h = g \circ f$ are skeletal, then so is g.*

Proof. Suppose for a contradiction that g fails to be skeletal. Then Y contains a non-empty open set V such that the image $g(V)$ is nowhere dense in Z. Since h is skeletal, the set $h^{-1}(g(V))$ is nowhere dense in X. It follows from $V \subset g^{-1}g(V)$ and $h^{-1}(g(V)) = f^{-1}g^{-1}(g(V))$ that the non-empty open set $f^{-1}(V)$ is nowhere dense in X. This contradiction completes the proof. $\square$

We know that open surjective mappings preserve the Baire property [12]. The same conclusion is valid for the wider class of skeletal mappings [13, Theorem 1]. For completeness sake we present a short proof of this fact.

Lemma 2.3. *Let $f\colon X \to Y$ be a continuous onto mapping. If f is skeletal and X is Baire, then Y is also Baire.*

Proof. Take a countable family $\{U_n : n \in \omega\}$ of open dense subsets of Y. Since f is skeletal and continuous, $V_n = f^{-1}(U_n)$ is a dense open subset of X, for each $n \in \omega$. The space X being Baire, the intersection $S = \bigcap_{n\in\omega} V_n$ is dense in X. It follows from our definition of the sets V_n's that $S = f^{-1}f(S)$, so $f(S) = \bigcap_{n\in\omega} U_n$ is dense in Y. Hence Y is Baire. $\square$

Suppose that $f\colon X \to Y$ and $g\colon X \to Z$ are continuous onto mappings. We write $f \prec g$ if there exists a continuous mapping $h\colon Y \to Z$ satisfying $g = h \circ f$. Let X be a space and $\mathcal{M}$ a family of continuous onto mappings of X elsewhere. We say that $\mathcal{M}$ is an *ω-directed lattice* for X if $\mathcal{M}$ generates the topology of X and for every countable subfamily $\mathcal{C}$ of $\mathcal{M}$, there exists $f \in \mathcal{M}$ such that $f \prec g$ for each $g \in \mathcal{C}$ (i.e. every countable subfamily of $\mathcal{M}$ has a lower bound in $\mathcal{M}$ with respect to the partial order $\prec$).

Proposition 2.4. *Let X be a space with an ω-directed lattice $\mathcal{M}$ of skeletal mappings and suppose that X has countable cellularity. Then X is Baire if and only if $f(X)$ is Baire, for each $f \in \mathcal{M}$.*

Proof. By Lemma 2.3, we have to prove the sufficiency only. Suppose that X is not Baire. It suffices to find an element $f \in \mathcal{M}$ such that $f(X)$ fails to be Baire. Notice first that the restriction of a skeletal mapping to an open subspace of X is again skeletal. Therefore, assuming that X is of the first category, we will find $f \in \mathcal{M}$ such that $f(X)$ is also of the first category. This requires the following simple fact.

Claim. *If A is a nowhere dense subset of X, then there exists $f \in \mathcal{M}$ such that $f(A)$ is nowhere dense in $f(X)$.*

Indeed, denote by $\mathcal{B}$ the family of sets of the form $g^{-1}(U)$, where $g \in \mathcal{M}$ and U is a non-empty open set in $g(X)$. Since $\mathcal{M}$ generates the topology of X, the family $\mathcal{B}$ is a base for X. Let γ_A be a maximal disjoint subfamily of $\mathcal{B}$ such that every element of γ_A is disjoint from A. By the maximality of γ_A, the open set $\bigcup \gamma_A$ is dense in $X \setminus A$ and, hence, in X. Since the cellularity of X is countable, we see that $|\gamma_A| \leq \omega$. Therefore, for every $V \in \gamma_A$, we can find $g_V \in \mathcal{M}$ and an open set $U_V \subset g_V(X)$ such that $V = g_V^{-1}(U_V)$. Since γ_A is countable, there is an element $f \in \mathcal{M}$ satisfying $f \prec g_V$ for each $V \in \gamma_A$. It is clear that $V = f^{-1}f(V)$ and $f(V)$ is open in $f(X)$, for each $V \in \gamma_A$. Hence $f(\bigcup \gamma_A)$ is a dense open set in $f(X)$. Since $A \cap \bigcup \gamma_A = \varnothing$, we infer that the sets $f(A)$ and $f(\bigcup \gamma_A)$ are disjoint, which in turn implies that $f(A)$ is nowhere dense in $f(X)$. This proves our claim.

Finally, suppose that $X = \bigcup_{n \in \omega} A_n$, where each A_n is nowhere dense in X. By our Claim, for every $n \in \omega$, there exists $f_n \in \mathcal{M}$ such that $f_n(A_n)$ is nowhere dense in $f_n(X)$. Take $f \in \mathcal{M}$ such that $f \prec f_n$ for each $n \in \omega$. It follows from our choice of f that for every $n \in \omega$, there exists a continuous mapping $g_n \colon f(X) \to f_n(X)$ satisfying $f_n = g_n \circ f$. By Lemma 2.2, the mappings g_n's are skeletal. Therefore, $\overline{f(A_n)}$ is nowhere dense in $f(X)$, for each $n \in \omega$—otherwise some of the sets $\overline{f_n(A_n)} = \overline{g_n(f(A_n))}$ would have a non-empty interior. This implies that the image $f(X)$ is of the first category and, hence, completes the proof. □

Corollary 2.5. *Let $\{Z_i : i \in I\}$ be a family of spaces such that the product $Z = \prod_{i \in I} Z_i$ has countable cellularity. Then a dense subspace X of Z is Baire if and only if $\pi_J(X)$ is Baire for each countable set $J \subset I$, where π_J is the projection of Z to $\prod_{i \in J} Z_i$.*

Proof. For every $J \subset I$, the projection π_J is an open surjective mapping. In particular, π_J is skeletal. Since X is dense in Z, the restriction of π_J to X, say, p_J is again skeletal. Notice that the family

$$\{p_J : J \subset I,\ |J| \leq \omega\}$$

is an ω-directed lattice for X. Therefore, the required conclusion follows from Proposition 2.4. □

Lemma 2.6. *Suppose that $f_i\colon X_i \to Y_i$ is a skeletal onto mapping, for each $i \in I$. Then the product mapping $\prod_{i\in I} f_i\colon \prod_{i\in I} X_i \to \prod_{i\in I} Y_i$ is also skeletal.*

Proof. First we prove the lemma in the special case of two mappings, say, f_1 and f_2. Let $g = f_1 \times f_2$ and take a non-empty open set U in $X_1 \times X_2$. We can find non-empty open sets U_1 and U_2 in X_1 and X_2, respectively, such that $U_1 \times U_2 \subset U$. Since f_1 and f_2 are skeletal, the set $\overline{f_i(U_i)}$ contains a non-empty open set V_i, for $i = 1, 2$. It is clear that

$$\overline{g(U)} \supset \overline{g(U_1 \times U_2)} = \overline{f_1(U_1)} \times \overline{f_2(U_2)}.$$

Hence the set $\overline{g(U)}$ contains the non-empty open set $V_1 \times V_2$ in $Y_1 \times Y_2$. This proves that $g = f_1 \times f_2$ is skeletal.

Applying induction, we see that the claim of the lemma is valid for finite products of skeletal mappings (notice that the surjectivity assumption has not been used so far).

Finally, we prove the lemma in the general case. Let $g = \prod_{i\in I} f_i$. It suffices to verify that $\overline{g(U)}$ has a non-empty interior, for every non-empty canonical open set U in $X = \prod_{i\in I} X_i$. Therefore, let

$$U = \bigcap_{i=1}^{n} \pi_{i_k}^{-1}(U_{i_k}),$$

where $i_1, \dots, i_n$ are pairwise distinct elements of the index set I, π_{i_k} is the projection of X onto X_{i_k}, and U_{i_k} is a non-empty open set in X_{i_k}, where $1 \le k \le n$. Since the mappings f_i's are surjective, it follows from the definition of g that

$$g(U) = \bigcap_{k=1}^{n} p_{i_k}^{-1}(D_k),$$

where p_{i_k} is the projection of $Y = \prod_{i\in I} Y_i$ onto the factor Y_{i_k} and $D_k = f_{i_k}(U_{i_k})$, $k = 1, \dots, n$. Denote by V_k the interior of the set $\overline{D}_k$ in Y_{i_k}, $1 \le k \le n$. Since the mappings f_{i_k}'s is skeletal, the sets $V_1, \dots V_n$ are non-empty. Hence the set $\overline{g(U)}$ contains the non-empty open set $\bigcap_{k=1}^{n} p_{i_k}^{-1}(V_k)$ in Y. So g is skeletal. □

Proposition 2.7. *Let $\{X_i : i \in I\}$ be a family of spaces such that every X_i has an ω-directed lattice of skeletal mappings onto separable (first countable, metrizable) spaces. Then the product space $X = \prod_{i\in I} X_i$ also has an ω-directed lattice of skeletal mappings onto separable (first countable, metrizable) spaces.*

Proof. For every $i \in I$, let $\mathcal{M}_i$ be an ω-directed lattice of skeletal mappings onto separable (first countable, or metrizable) spaces. For every countable set $J \subset I$, put

$$\mathcal{M}_J = \left\{ \prod_{i\in J} f_i : f_i \in \mathcal{M}_i \text{ for each } i \in J \right\}.$$

By Lemma 2.6, the family $\mathcal{M}_J$ consists of skeletal surjective mappings.

For every $J \subset I$, denote by π_J the natural projection of X onto the subproduct $X_J = \prod_{i \in J} X_i$. Clearly the projections π_J's are open, continuous, and surjective. Hence the family

$$\mathcal{M} = \{f \circ \pi_J : J \subset I,\ |J| \leq \omega,\ f \in \mathcal{M}_J\}$$

consists of skeletal mappings of X onto separable (first countable, metrizable) spaces. Since each $\mathcal{M}_i$ is an ω-directed lattice, it follows from our definitions that so is $\mathcal{M}$. □

Suppose that X is a space with an ω-directed lattice $\mathcal{M}$ of mappings of X elsewhere. Then $\mathcal{M}$ is said to be a *σ-lattice* for X if for every sequence $\{f_n : n \in \omega\}$ in $\mathcal{M}$, the diagonal product $\Delta_{n \in \omega} f_n$ of the mappings f_n's is also an element of $\mathcal{M}$.

A typical example of a σ-lattice for a subspace X' of a product $X = \prod_{i \in I} X_i$ comes if we consider consider the family of restrictions to X' of projections of X to countable subproducts.

Proposition 2.8. *If a space X has a σ-lattice of skeletal mappings onto spaces of countable cellularity, then X also has countable cellularity.*

Proof. Let $\mathcal{M}$ be a σ-lattice of skeletal mappings of X onto spaces of countable cellularity. Since $\mathcal{M}$ generates the topology of X, the family

$$\mathcal{B} = \{f^{-1}(U) : f \in \mathcal{M},\ U \text{ is open in } f(X),\ \text{and } U \neq \varnothing\}$$

forms a base for X.

Consider a disjoint family γ of non-empty open sets in X. We can assume without loss of generality that $\gamma \subset \mathcal{B}$ and that γ is maximal with respect to the inclusion relation. Hence $\bigcup \gamma$ is dense in X. Take an arbitrary element $V_0 \in \gamma$ and put $\gamma_0 = \{V_0\}$. Since $V_0 \in \gamma \subset \mathcal{B}$, we can find $f_0 \in \mathcal{M}$ and a non-empty open set U_0 in $f(X)$ such that $V_0 = f_0^{-1}(U_0)$. Let $\mathcal{C}_0 = \{f_0\}$.

Suppose that for some $n \in \omega$, we have defined subfamilies $\gamma_0, \ldots, \gamma_n$ of γ and elements $f_0, \ldots, f_n$ of $\mathcal{M}$ satisfying the following conditions for all k, m with $0 \leq k < m \leq n$:

(i) $|\gamma_m| \leq \omega$;
(ii) $f_k \prec f_m$;
(iii) every $V \in \gamma_m$ has the form $f_m^{-1}(U)$, for a non-empty open set U in $f_m(X)$;
(iv) the set $f_k(\bigcup \gamma_{k+1})$ is dense in $f_k(X)$.

For every $V \in \gamma$, put $U_V = \operatorname{int} \overline{f_n(V)}$. By Lemma 2.1, U_V is dense in $\overline{f_n(V)}$, for each $V \in \gamma_n$. Since $\bigcup \gamma$ is dense in X and f_n is continuous, the union of the family $\{V_U : V \in \gamma\}$ is dense in $X_n = f_n(X)$. By our assumptions, the space X_n has countable cellularity. Hence there exists a countable subfamily γ_{n+1} of γ such that $\bigcup_{V \in \gamma_{n+1}} V_U$ is dense in X_n. For every $V \in \gamma_{n+1}$, take an element $f_V \in \mathcal{M}$ and a non-empty open set U_V in $f_V(X)$ such that $V = f_V^{-1}(U_V)$. Since the family $\{f_V : V \in \gamma_{n+1}\}$ is countable, there exists $f_{n+1} \in \mathcal{M}$ such that $f_{n+1} \prec f_n$ and $f_{n+1} \prec f_V$ for each $V \in \gamma_{n+1}$. Hence the families

$\gamma_0, \dots, \gamma_n, \gamma_{n+1}$ and elements $f_0, \dots, f_n, f_{n+1}$ of $\mathcal{M}$ satisfy (i)–(iv) at the step $n+1$.

It follows from (i) that $\gamma_* = \bigcup_{n\in\omega} \gamma_n$ is a countable subfamily of γ. Since $\mathcal{M}$ is a σ-lattice for X and $f_{n+1} \prec f_n \in \mathcal{M}$ for each $n \in \omega$, we can assume that the diagonal product of the family $\{f_n : n \in \omega\}$, say, f_* is an element of $\mathcal{M}$. Then (iii) implies that $V = f_*(V)$ is open in $X_* = f_*(X)$ and $V = f_*^{-1} f_*(V)$, for each $V \in \gamma_*$. It also follows from our choice of $f_* \in \mathcal{M}$ and condition (iv) that the open set $f_*(\bigcup \gamma_*)$ is dense in X_*. Since f_* is skeletal, the set $\bigcup \gamma_* = f_*^{-1} f_*(\bigcup \gamma_*)$ is dense in X. However, the family γ is disjoint and $\gamma_* \subset \gamma$, whence it follows that $\gamma_* = \gamma$. We conclude that $|\gamma| \leq \omega$ and, hence, the cellularity of X is countable. □

Combining Propositions 2.4 and 2.8, we obtain the following result:

Theorem 2.9. *If a space X has a σ-lattice of skeletal mappings onto Baire spaces of countable cellularity, then X is Baire as well.*

For a Tychonoff space X, let $C_p(X)$ be the space of continuous real-valued functions on X endowed with the pointwise convergence topology. It was shown by Tkachuk in [25, Theorem 4.8] that the product of an arbitrary family of spaces of the form $C_p(X)$ is Baire provided that each factor is Baire. Since $C_p(X)$ is dense in $\mathbb{R}^X$, where $\mathbb{R}$ is the real line, Tkachuk's theorem follows from the next result:

Corollary 2.10. *Let $\{X_i : i \in I\}$ be a family of Baire spaces, where each X_i is a dense subspace of a product of regular spaces of countable π-weight. Then the product space $X = \prod_{i\in I} X_i$ is also Baire.*

Proof. It is clear that X is a dense subspace of a product of spaces with countable π-bases, say, $Y = \prod_{j\in J} Y_j$. Taking the restriction to X of the projections of Y to countable subproducts, we obtain a σ-lattice of skeletal mappings of X onto spaces with countable π-bases (notice that a dense subspace of a space with a countable π-base also has a countable π-base). It is also clear that every space with a countable π-base has countable cellularity. By Oxtoby's theorem in [18], every (countable) subproduct of Y is Baire. Hence X has a σ-lattice of skeletal mappings onto Baire spaces of countable cellularity. The required conclusion now follows from Theorem 2.9. □

3. Products of paratopological groups

In independent papers, Oxtoby [18] and Frolík [12] introduced a number of classes of Baire spaces, including as examples the class of Čech-complete spaces and the class of regular feebly compact spaces, i.e. those where locally finite families of open subsets are finite. We also recall that a space X is *countably complete* (*pseudo-complete*) if there exists a sequence of (pseudo-) bases $\{\mathcal{B}_n\}$ for X such that for each decreasing sequence of open sets $\{U_n\}$ in X, where each U_n is contained in some element of $\mathcal{B}_n$, we have $\bigcap U_n \neq \varnothing$. It was proved in [18, 12] that if $\{X_\alpha\}_{\alpha\in A}$ is a family of spaces from any of the aforementioned classes, then the product $X = \prod_{\alpha\in A} X_\alpha$ also belongs to the same class, so X is a Baire space.

On the other hand, according to the well-known theorem of Comfort and Ross [10], the product of an arbitrary family of pseudocompact topological groups is pseudocompact. Hence such products have the Baire property. It is also known that pseudocompact topological groups are precompact [10, Theorem 1.1]. It was recently proved in [7] that the class of Baire precompact topological groups is closed under taking arbitrary products. Now we extend the latter result to the class of Baire precompact *paratopological* groups, i.e. the groups with jointly continuous multiplication.

Suppose that (G, τ) is a paratopological group. Let us denote by τ_* the finest topological group topology coarser than τ. Also, let G_* be the corresponding topological group (G, τ_*). We need the following fact proved in [24, Theorem 10]:

Lemma 3.1. *Let $\{G_i\}_{i\in I}$ be a family of paratopological groups. Then $G_* \cong \prod_{i\in I}(G_i)_*$.*

In general, given two topologies τ_1 and τ_2 on a set X with $\tau_2 \subset \tau_1$, the Baire property of $X_1 = (X, \tau_1)$ does not imply that $X_2 = (X, \tau')$ is Baire nor vice versa. In the next lemma we find an additional condition on τ_1 and τ_2 which is responsible for the preservation of the Baire property in both directions.

Lemma 3.2. *Let τ_1 and τ_2 be two topologies on a set X such that $\tau_2 \subset \tau_1$. If τ_2 is a π-base for X_1, then the families of nowhere dense sets in X_1 and X_2 coincide. In particular, X_2 is a Baire space if and only if so is X_1.*

Proof. Given a nowhere dense set $F \subset X_1$ and a non-empty open set $O_2 \subset X_2$, there exists a non-empty open set $O_1 \subset X_1$ such that $O_1 \subset O_2$ and $O_1 \cap F_1 = \varnothing$. Let $U_2 \subset X_2$ be a non-empty open set contained in O_2. Then $U_2 \subset O_2$ and $U_2 \cap F_1 = \varnothing$, so F_1 is nowhere dense in X_2. Conversely, if F_2 is nowhere dense in X_2, then the inclusion $\tau_2 \subset \tau_1$ implies that F_2 is nowhere dense in X.

Now, if a non-empty open set $U_1 \subset X_1$ is of the first category in X_1, then a non-empty open set $U_2 \subset X_2$ contained in U_1 is also of the first category in X_2. □

Remark. It follows from Lemma 3.2 that the closure in X_2 of every nowhere dense subset of X_1 is nowhere dense in X_2.

We recall that a paratopological group G is called *saturated* if U^{-1} has a non-empty interior in G, for every neighborhood U of the neutral element in G. The Sorgenfrey line is an example of a saturated paratopological group which is not a topological group. It is easy to see that an arbitrary product of saturated paratopological groups is again saturated.

Theorem 3.3. *Let $\{G_i\}_{i\in I}$ be a family of Baire precompact paratopological groups. Then $G = \prod_{i\in I} G_i$ is a Baire precompact paratopological group.*

Proof. Since each G_i is Baire and precompact, it is saturated by [20, Proposition 3.1] or [2, Theorem 2.5]. Hence the topology of $(G_i)_*$ forms a π-base for G_i (see [4, Theorem 5]). Apply Lemma 3.1 to conclude that $G_* \cong \prod_{i\in I}(G_i)_*$, where each $(G_i)_*$ is a Baire precompact topological group by Lemma 3.2. Hence, according to [4, Theorem 3.4], G_* is a Baire precompact topological group. Since G is precompact and the topology of G_* is a π-base for G, Lemma 3.2 implies that G is Baire. □

The Comfort-Ross theorem on products of pseudocompact topological groups was extended by Ravsky [21] to products of feebly compact paratopological groups. Furthermore, Ravsky noted that every feebly compact paratopological group satisfying the T_3 separation axiom was a topological group, thus weakening the regularity assumption in a theorem of Arhangel'skii and Reznichenko [3, Theorem 1.7]. On the other hand, Sanchis and Tkachenko [22] constructed the first examples of Hausdorff feebly compact paratopological groups with the Baire property that were not topological groups. This makes it tempting to prove or refute the following conjecture.

Conjecture 3.4. *Let $\{G_i\}_{i\in I}$ be a family of Baire feebly compact paratopological groups. Then $G = \prod_{i\in I} G_i$ is a Baire feebly compact paratopological group.*

4. Topological games and the Baire property

In this section we apply the topological game methods to handle a number of complete-type topological spaces close to the Baire ones. The so-called Banach-Mazur-Choquet game or *Choquet game*, for brevity, $\mathcal{G}(X)$ is played on a space (X, τ) by players α and β. Both players alternatively choose elements from $\tau^* = \tau \setminus \{\varnothing\}$ as follows.

i) Player β moves first choosing a set $B_1 \in \tau^*$.
ii) Player α then responds by choosing a set $A_1 \in \tau^*$ contained in B_1.
iii) At the n-th step player β chooses a set $B_n \in \tau^*$ contained in A_{n-1}.
iv) Player α responds by choosing a set $A_n \in \tau^*$ contained in B_n.

The sequence

$$B_1 \supset A_1 \supset B_2 \supset A_2 \supset \dots$$

obtained according to i)–iv), also denoted by $\{(A_n, B_n)\}_n$, is called a *play*. Player α *wins* the play $\{(A_n, B_n)\}_n$ if $\bigcap_{n\in\mathbb{N}} A_n \neq \varnothing$, otherwise β wins the play.

By a strategy for a player we mean a rule that specifies each move of the player in every possible situation. More precisely, a strategy $t = \{t_n : n \in \mathbb{N}\}$ for β consists of a sequence of τ^*-valued mappings such that

1. $t_1 \colon \{\varnothing\} \to \tau^*$, $B_1 = t_1(\varnothing)$;
2. $t_n \colon \{(A_1, \dots, A_{n-1}) \in (\tau^*)^{n-1} : A_j \subset t_j(A_1, \dots, A_{j-1}),\ 1 \le j \le n-1\} \to \tau^*$, $B_n = t_n(A_1, \dots, A_{n-1})$.

Such a sequence of subsets $\{A_n\}_n$ is called a *t-play*. A strategy t is called a *winning strategy* for player β if β wins each t-play. Strategies and winning strategies for player α are defined similarly. The importance of this type of games is based, in particular, on the following characterization of Baire spaces.

Theorem 4.1 (Banach-Mazur-Choquet). *A space X is Baire if and only if player β does not have a winning strategy in the Choquet game $\mathcal{G}(X)$.*

In particular, a space with a winning strategy for player α, or α-*favorable* space, is a Baire space. For example, a countably complete space is α-favorable. The class of α-favorable spaces is different from the class of β-*unfavorable* spaces or Baire spaces. In fact, this class is closed under arbitrary products, as it was proved by White in [29].

Using topological games, one can define new classes of topological spaces. Given a dense subset $D \subset X$, the game $\mathcal{G}_S(D)$ played on X by two players, α and β, has the same playing rules as the Choquet game, with the only difference in the winner decision rule. We declare that the player α wins a play $\{(A_n, B_n)\}_n$ provided that

(1) $\bigcap A_n \neq \varnothing$;
(2) every sequence $\{a_n\}_n$ with $a_n \in A_n \cap D$, for each $n \in \mathbb{N}$, has an accumulation point in X.

Following Kenderov, Kortezov, and Moors [15], we say that a regular space X is *strongly Baire* if there exists a dense subset D of X such that player β does not have a winning strategy in the $\mathcal{G}_S(D)$-game played on X. In [15], the authors extended the results from [6] on the continuity of the group operations in Čech-complete semitopological groups to the wider class of strongly Baire spaces. The following theorem is the main result established in [15].

Theorem 4.2 (Kenderov-Kortezov-Moors). *Let G be a regular semitopological group. If G is strongly Baire, then G is a topological group.*

Since there exist regular Baire paratopological groups that are not topological groups (the Sorgenfrey line is an example), Theorem 4.2 distinguishes the classes of Baire and strongly Baire paratopological groups. Further, as in the case of Baire spaces, it is easy to see that the class of strongly Baire spaces is closed under taking open subsets, dense G_δ-subsets, and images of continuous open mappings. Moreover, a space having a dense strongly Baire subspace is itself strongly Baire. We will show that some generalized metric Baire spaces are, in fact, strongly Baire.

Let us recall that a Hausdorff space X is called a *Lindelöf Σ-space* if it has a cover $\mathcal{C}$ by compact sets and a countable family $\mathcal{F}$ of closed subsets such that for each $C \in \mathcal{C}$ and each open set U containing C, there exists $F \in \mathcal{F}$ such that $C \subset F \subset U$.

Notice that for a fixed $x \in X$, if $\{F_n\}_n$ is an enumeration of $\mathcal{F}(x) = \{F \in \mathcal{F} : x \in F\}$, then every sequence $\{x_n\}_n$ such that $x_n \in F_n$ for each $n \in \mathbb{N}$, has an accumulation point in X.

Lemma 4.3. *Every regular, Baire, Lindelöf Σ-space X is strongly Baire.*

Proof. Let $\mathcal{C}$ be a cover of X by compact sets and $\mathcal{F} = \{F_n : n \in \mathbb{N}\}$ a countable family of closed subsets of X witnessing that X is a Lindelöf Σ-space. Let also $t = \{t_n\}_n$ be a strategy for player β in the game $\mathcal{G}_S(X)$. We will prove that there exists a t-sequence $\{A_n\}_n$ such that player α wins the t-play $\{A_n\}_n$. For this purpose, we define inductively a new strategy $t' = \{t'_n\}$ for player β as follows. In the first play, we use $t'_1(\varnothing) = t_1(\varnothing)$. Suppose that we have already defined functions $t'_1, \dots, t'_{n-1}$ in such a way that each partial t'-sequence $(A_1, \dots, A_{j-1})$ is also a partial t-sequence ($j \leq n-1$). Then we define a function t'_n by the rule

$$t'_n(A_1, \dots, A_{n-1}) = \begin{cases} t_n(A_1, \dots, A_{n-1}) \cap \operatorname{int} F_n, & \text{if this set is non-empty;} \\ t_n(A_1, \dots, A_{n-1}) \setminus F_n, & \text{otherwise.} \end{cases}$$

Since X is Baire, there exists a t'-sequence $\{A_n\}_n$ such that the intersection $\bigcap_{n\in\mathbb{N}} A_n$ is not empty. Then $\{A_n\}_n$ is also a t-sequence and we claim that this t-sequence witnesses that player α wins in $\mathcal{G}_S(X)$.

Indeed, let $\{a_n\}_n$ be a sequence such that $a_n \in A_n$, for each $n \in \mathbb{N}$. Pick a point $x \in \bigcap_{n\in\mathbb{N}} A_n$ and a sequence $\{n_k\}_k$ in $\mathbb{N}$ such that $\mathcal{F}(x) = \{F_{n_k}\}_k$. Then, by the definition, $a_{n_k} \in A_{n_k} \subset B'_{n_k} \subset \operatorname{Int} F_{n_k}$, for each $k \in \mathbb{N}$. So the subsequence $\{a_{n_k}\}_k$ of $\{a_n\}_n$ has an accumulation point in X. □

Combining Theorem 4.2 and Lemma 4.3, we obtain the following fact:

Corollary 4.4. *Let G be a regular Baire semitopological group. If G is a Lindelöf Σ-space, then G is a topological group.*

According to Hodel [14], a space X is called a *$w\Delta$-space* if X has a sequence $\{\mathcal{G}_n\}_n$ of open covers such that for every $x \in X$, if $\{x_n\}_n$ is a sequence with $x_n \in \operatorname{St}(x, \mathcal{G}_n)$ for each $n \in \mathbb{N}$, then $\{x_n\}_n$ has an accumulation point in X.

Lemma 4.5. *Every regular Baire $w\Delta$-space X is strongly Baire.*

Proof. Let $\{\mathcal{G}_n\}_n$ be a sequence of open covers of X witnessing that X is a $w\Delta$-space. Suppose that $t = \{t_n\}_n$ is a strategy for player β in the game $\mathcal{G}_S(X)$. Again, we define a new strategy $t' = \{t'_n\}_n$, just as we did it in the proof of Lemma 4.3, using the slightly different rule

$$t'_n(A_1, \dots, A_{n-1}) = t_n(A_1, \dots, A_{n-1}) \cap G_n,$$

where $G_n \in \mathcal{G}_n$ is any element intersecting $t_n(A_1, \dots, A_{n-1})$.

According to this definition, the Baire property of X implies the existence of a t'-sequence $\{A_n\}_n$ such that $\bigcap_{n\in\mathbb{N}} A_n \neq \varnothing$. Now we choose a sequence $\{a_n\}_n$, where $a_n \in A_n$ for each $n \in \mathbb{N}$. Pick a point $x \in \bigcap A_n$. Note that

$$\{x, a_n\} \subset A_n \subset B'_n \subset G_n \subset \operatorname{St}(x, \mathcal{G}_n),$$

for each n. Hence $\{a_n\}_n$ has an accumulation point in X and $\{A_n\}_n$ is a t-sequence for player α in $\mathcal{G}_S(X)$. So α wins the t-play $\{A_n\}_n$ in the game $\mathcal{G}_S(X)$. □

Corollary 4.6. *Let G be a regular Baire semitopological group. If G is $w\Delta$-space, then G is a topological group.*

Since every Moore space is a $w\Delta$-space [14], the fact below is immediate from Corollary 4.6.

Corollary 4.7 (Piotrowski, [19])**.** *Let G be a Baire semitopological group. If G is a Moore space, then G is a topological group.*

As metrizable spaces are Moore spaces, every metrizable Baire space is automatically a strongly Baire one. It was proved by van Douwen in [28] that every Baire σ-space contains a dense metrizable G_δ-subset. This implies that all Baire σ-spaces are also strongly Baire.

5. Open problems

In view of the importance of strongly Baire spaces we formulate several problems about them, whose solutions can help for better understanding the properties of this class.

Problem 5.1. *Let X and Y be strongly Baire (metric) spaces or topological groups.*

(a) *When is $X \times Y$ strongly Baire?*
(b) *Is the product $X \times Y$ strongly Baire provided it is a Baire space?*
(c) *What if Y is additionally hereditarily Baire?*

In all known examples showing that the Baire property is not productive, the corresponding spaces are far from separable. This fact motivates the problem below.

Problem 5.2. *Do there exist separable (regular, Tychonoff) Baire spaces X and Y such that the product $X \times Y$ fails to be Baire?*

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