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Hausdorff connectifications

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Abstract

Disconnectedness in topological space is analyzed to obtain Hausdorff connectifications of that topological space. Hausdorff connectifications are obtained by some direct constructions and by some partitions of connectifications. Also lattice structure is included in the collection of all Hausdorff connectifications.

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1. Introduction

An extension of a topological space X is a topological space that contains X as a dense subspace. If an extension is a connected space, then that extension is called a connectification of that space. There is a book of J. R. Porter and R. Grandwoods that is devoted for Hausdorff extensions. This paper also studies only Hausdorff spaces and Hausdorff connected extensions. It is easy to see that if X has a proper compact (or H-closed) open subset, then X has no Hausdorff connectification. There are spaces which can not have connectification. Porter and Grandwoods gave some nice examples of Hausdorff spaces in [7], that can not be densely embedded in a connected Hausdorff space. Several papers have been devoted to connectifications (See: [1], [2], [3], [9]). Fedeli and Le Donne in [4] proved that a T_1-space can be densely embedded in a pathwise connected T_1-space if and only if it has no isolated points. Charatonik in [3] considered generalized linear graphs to obtain Hausdorff connectifications and characterized the one point Hausdorff connectification of a subspace of a generalized

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linear graph in [2]. Section 2 contains some basic ideas about disconnectedness of topological spaces. section 3 presents some direct constructions to obtain Hausdorff connectifications of a topological space. We also obtain Hausdorff connectifications through remainders in section 4. Final section proves that if f is a continuous and connected mapping from X onto Y such that f seperates every pair of disjoint regular open subsets of X, then the lattice $\mathcal{C}(X)$ is isomorphic to $\mathcal{C}(Y)$. All spaces under consideration are Hausdorff topological spaces.

2. Some Disconnected Spaces

Definition 2.1. A subset A of a space X is n-disconnected if A has exactly $n+1$ no. of clopen subsets in A except $\varnothing$ and A.

Definition 2.2. A subset A of a space X is countably infinite disconnected if A has only countably infinite number of clopen subsets in A. A is countably disconnected if it is either n-disconnected or countably infinite disconnected.

Definition 2.3. A subset A of a space X is uncountably disconnected if A is not countably disconnected.

Example 2.4. (*i*) $(0,1)\cup(1,2)$ is 1-disconnected.

(*ii*) $\bigcup\limits_{k=1}^{n}(k,k+1)$ is (2^n-2)-disconnected.

(*iii*) Set of all irrationals is uncountably disconnected subset of R.

Theorem 2.5. *Let $f : X(\subseteq Z_1) \to Z$ be a continuous mapping such that $f(X) = Y \subseteq Z$. If Y is n-disconnected subset of Z, then X is atleast a n-disconnected subset of Z_1. Also, if A is a component in Y, then $f^{-1}(A)$ is a component in X.*

Proof. Let Y be a n-disconnected subset of a space Z. Then Y has $n+1$ no of clopen subsets. Let them be $\{A_1, A_2, \cdots\cdots A_{n+1}\}$. Since f is continuous, X has atleast $n+1$ no of clopen subsets namely, $\{B_1, B_2, \cdots\cdots B_{n+1}\}$. Also, if A is a component in Y, then $f^{-1}(A)$ is a component in X. If not, then there is a connected subset C of X containing $f^{-1}(A)$. Then $f(C)$ is a connected subset of Y containing A, which is a contradiction to the maximality of A. □

Theorem 2.6. *Let $f : X \to Z$ be an one to one and open mapping. If $Y = f(X)$ and if X is a n-disconnected subset of Z_1, then Y is atleast a n-disconnected subset of Z. Also, image of a component under the mapping f is a component in Y.*

Proof. If f is an one to one and open mapping, then $f^{-1} : Y \to X$ is a continuous mapping. By the previous theorem 2.5, X is a n-disconnected subset of Z_1. □

A subspace of a n-disconnected space need not be a n-disconnected space. Consider a subspace $A = [0,1]\cup[2,3]$ of $[0,1]\cup[2,3]\cup[4,5]$, for some fixed n. Then A is 1-disconnected subspace of a 2-disconnected space $[0,1]\cup[2,3]\cup[4,5]$.

Theorem 2.7. *Let A be subset of a n-disconnected space X. Then A is n-disconnected if A is a connected dense subspace of X.*

Definition 2.8. A point $x \in Y$ is a cut point of $X \subseteq Y$ if there is a clopen subset A of X such that $A \cup \{x\} \cup (X \setminus A)$ is a connected subset of Y.

Theorem 2.9. *If $X (\subseteq Y)$ has n cut points, then X is atleast a $n+1$-disconnected subspace of Y.*

Proof. If X has n cut points (say) $p_1, p_2, \cdots\cdots p_n$, then there are n-clopen subsets $\{A_i : i = 1, 2, \cdots\cdots n\}$ such that $A_i \cup (X \setminus A_i) \cup \{p_i\}$ is a connected subset of Y. Thus X has $2n$ clopen subsets. Also unions and intersections of clopen subsets are clopen subsets which increases the no of clopen subsets of X. Thus X is atleast a $n+1$-disconnected subspace of Y. □

There may be a clopen subset A_k of X such that $A_k \cup (X \setminus A_k) \cup E$ is a connected subset of Y, where E is a subset of Y and E contains more than one point. This also increases the no of clopen subsets of X.

Theorem 2.10. *Let X be a subspace of a space Y. If X is n-disconnected subspace of Y, then X has atmost n cut points.*

Theorem 2.11. *A point $x \in Y$ is a cut point of $X \subseteq Y$ if and only if there is a connected subset C of Y such that $C \cap X = A \cup (X \setminus A)$ and $Y \setminus (C \cap X) = \{x\}$, where A is a clopen subset of X.*

Proof. Proof follows directly from the definition of cut point. □

3. Some Connectifications

Theorem 3.1. *Let X be a space having no isolated points. If X has finite number of distinct clopen subsets(2n-disconnected space) such that each clopen subset is neither H-closed nor compact, then there is an extension Y of X such that Y is connected and hence Y is a connectification of X.*

Proof. Let X be a space having $2n$ distinct clopen subsets of X. Let them be $\{A_1, A_2, \cdots\cdots A_{2n}\}$. Let $A_{11} = A_1$. Since $X \setminus A_1$ is a clopen subset of X, $X \setminus A_1$ is one of the member in $\{A_2, \cdots\cdots A_{2n}\}$. Let it be A_{12}. Let $A_{21} \in \{A_1, A_2, \cdots\cdots A_{2n}\} \setminus \{A_{11}, A_{12}\}$. Then $X \setminus A_{21} \in \{A_1, A_2, \cdots\cdots A_{2n}\} \setminus \{A_{11}, A_{12}, A_{21}\}$. Let it be A_{22}. Thus we can arrange the clopen subsets $\{A_1, A_2, \cdots\cdots A_{2n}\}$ as $\{A_{11}, A_{12}, A_{21}, A_{22} \cdots \ \cdots A_{n1}, A_{n2}\}$. Let $Y = X \cup \{p_1, p_2, \cdots\cdots p_n\}$, where $p_i \notin X$ for every $i \in \{1, 2, \cdots\cdots n\}$. Define the open neighborhoods of p_i be $U_{i1} \cup U_{i2} \cup p_i$, where $U_{ij} \in \mathcal{U}_{ij}$ and $\mathcal{U}_{ij}$ is a decreasing sequence of nonempty open subsets of A_{ij}'s. Then Y is an extension of X. If Y is not connected, then there exists a clopen subset C of Y such that $C \cap X$ is a clopen subset of X. By construction, there are some A_{i1} and A_{i2} such that $C \cap X = A_{i1}$ or $C \cap X = A_{i2}$. Then there is an element $p_i \in Y$ such that every open neighborhood of p_i intersects $C \cap X$ and $X \setminus C$. Thus $p_i \in cl_Y(C \cap X)$ and $p_i \in cl_Y(X - C)$. That is $p_i \in cl_Y(C) = C$ and $p_i \in cl_Y(Y - C) = Y - C$. That is $p_i \in C \cap (Y - C)$, which is a contradiction. It remains only to prove that Y is a Hausdorff space. Let $x, y \in Y$ and $x \neq y$.

Case: 1. Let $x, y \in Y \backslash X$. Then there exists p_i and p_j such that $x = p_i$ and $y = p_j$. We can find nonempty open subsets U_{i1}, U_{i2}, V_{j1} and V_{j2} of A_{i1}, A_{i2}, A_{j1} and A_{ji2}, respectively such that $U_{i1} \cup U_{i2} \cup \{p_i\}$ and $V_{j1} \cup V_{j2} \cup \{p_j\}$ are open sets in Y containing x and y, respectively. Let $x_1 \in U_{i1}$, $x_2 \in U_{i2}$, $x_3 \in V_{j1}$ and $x_4 \in V_{j2}$. Find disjoint open neighborhoods $W_{x_1}, W_{x_2}, W_{x_3}, W_{x_4}$ for x_1, x_2, x_3, x_4 such that $W_{x_1} \subseteq U_{i1}$, $W_{x_2} \subseteq U_{i2}$, $W_{x_3} \subseteq V_{j1}$ and $W_{x_4} \subseteq V_{j2}$. Let $U = W_{x_1} \cup W_{x_2} \cup \{p_i\}$ and $V = W_{x_3} \cup W_{x_4} \cup \{p_j\}$. Thus U and V are disjoint open subsets of p_i and p_j, respectively.

Case: 2. Let $x \in X$ and $y \in Y \backslash X$. Then there exists a clopen subset A_i in X and $p_i \in Y$ such that $y = p_i$ and $x \in A_{i1}$ or $x \in A_{i2}$. Let us assume that $x \in A_{i1}$. Choose any open neighborhood $U_{1x} \cup U_{2x} \cup \{p_i\}$ of p_i and $x_0 \in U_{1x}$ such that $x \neq x_0$. Then there are disjoint open sets $V_x \subseteq U_{1x}$ and $V_{x_0} \subseteq U_{1x}$ such that $x \in V_x$ and $x_0 \in V_{x_0}$. Let $W = V_{x_0} \cup W_{x_0} \cup \{p_i\}$, where W_{x_0} is any nonempty open subset in A_{i2}. Then U_x and W are disjoint open neighborhoods of x and y, respectively.

Case: 3. Let $x, y \in X$. Then there are disjoint open neighborhoods for x and y in X. By our construction, They are also open in Y. □

If X is finitely disconnected and has no isolated points, then we can give a one point connectification of X. we can attach only one point, whose open neighborhoods are of the form $U_1 \cup U_2 \cup \cdots \cdots U_n$, where $U_i \in \mathcal{U}_i$ and $\mathcal{U}_i$ is decreasing sequence of non empty open subsets of the clopen subset A_i of X.

Corollary 3.2. *If X is a space having countable number of distinct clopen subsets of X (countably disconnected space) such that each clopen subset is neither H-closed nor compact and having no isolated points, then there is an extension Y of X such that Y is connected.*

Corollary 3.3. *If X is a space having uncountable number of distinct clopen subsets of X (uncountably disconnected space) such that each clopen subset is neither H-closed nor compact and having no isolated points, then there is an extension Y of X such that Y is connected.*

Definition 3.4. Let $f : X \to Y$ be a mapping from X to Y. Then f is said to separates disjoint subsets A and B of X if $f(A) \cap f(B) = \phi$.

Theorem 3.5. *Let $f : X \to Y$ be a continuous mapping from X onto Y such that f seperates every pair of disjoint regular open subsets of X. If ξX is any connectification of X, then there is a connectification ζY for Y.*

Proof. Let $\zeta Y = Y \cup K$, where $K = \xi X \backslash X$. Let U be an open set in Y, then $f^{-1}(U)$ is open in X. Find an open set V in ξX such that $V \cap X = f^{-1}(U)$. Let $W = V \cap X = f^{-1}(U)$ and $W_1 = V \cap K$. Define a topology on ζY by the collection of all sets of the form $f(W) \cup W_1$, where $W = V \cap X$, $W_1 = V \cap K$ and $V = f^{-1}(U)$ is an open subset of ξX. We prove that ζY with this topology is a connectification of Y.

Let V be any open set in ζY. Then there exist an open set U in ξX such that $V = f(U \cap X) \cup (U \setminus X)$. We may choose $U = f^{-1}(V) \cap Y$. Since X is dense in ξX, $U \cap X \neq \phi$. Thus $f(U \cap X) \cap Y \neq \phi$ and hence $V \cap Y \neq \phi$. This implies that Y is dense in ζY.

If ζY is not connected, there exists a nonempty clopen subset U of ζY such that $U \subset \zeta Y$ and $U \neq \zeta Y$. Then, there exists an open set $V(= f^{-1}(U \cap Y))$ such that $f(V) = U \cap Y$. Then $V \cup H$ is an open set in ξX, where $H = U \setminus Y$. Similarly, since $Y \setminus U$ is an open set, there exists an open set $V_1(= f^{-1}((Y \setminus U) \cap Y))$ such that $f(V_1) = Y \setminus U$. Then $V_1 \cup H_1$ is an open set in ξX, where $H_1 = (Y \setminus U) \setminus Y$. Trivially, $\xi X \setminus (V \cup H) = V_1 \cup H_1$. Also $V \cup H$ and $V_1 \cup H_1$ are both open sets. This proves that ζY is connected.

It remains only to prove that ζY is a Hausdorff space. Let $x, y \in Y$ and $x \neq y$. If x is in K, let $x_1 = x$ and if y is in K, let $y_1 = y$. Otherwise choose $x_1 \in f^{-1}(x)$ and $y_1 \in f^{-1}(y)$. Then there are disjoint open sets U and V in ξX such that $x_1 \in U$, $y_1 \in V$ and $int(cl(U)) \cap int(cl(V)) = \phi$. Let $U_1 = int(cl(U)) \cap X$, $U_2 = int(cl(U)) \setminus X$, $V_1 = int(cl(V)) \cap X$ and $V_2 = int(cl(V)) \setminus X$. Since f seperates every pair of disjoint regular open subsets in X, we have $f(U_1) \cap f(V_1) = \phi$. Then $f(U_1) \cup U_2$ and $f(V_1) \cup V_2$ are disjoint open subsets of ζY containing x and y, respectively. □

Remark 3.6. In the above theorem 3.5, the condition "f seperates every pair of disjoint regular open subsets of X" used only for Hausdorffness of a connectification. Without that assumption, we can get a non Hausdorff connectification of a space Y.

Definition 3.7 ([8])**.** A mapping $f : X \to Y$ is a said to be compact mapping if $f^{-1}(x)$ is a compact subset of X, for everey $x \in X$.

Theorem 3.8. *Let $f : X \to Y$ be a continuous and compact mapping from X onto Y. If ξX is any connectification of X, then there is a connectification ζY for Y*

Proof. Proof for "ζX is a connected extension of X" follows from theorem 3.5. Hausdorffness of ζX is to be proved. Let $x, y \in Y$ and $x \neq y$. $K = \xi X \setminus X$. If x is in K, let $C_1 = \{x\}$ and if y is in K, let $C_2 = \{y\}$. Since f is a compact maping, $f^{-1}(x) = C_1$ and $f^{-1}(y) = C_2$ are compact subsets of X. Then there are disjoint open sets U and V in ξX such that $C_1 \subseteq U$ and $C_2 \subseteq V$. Then $f(U \cap X) \cup (U \setminus X)$ and $f(V \cap X) \cup (V \setminus X)$ are disjoint open subsets of ζX containing x and y respectively. □

Definition 3.9. A mapping $f : X \to Y$ is said to be a pointly connected mapping if $f^{-1}(x)$ is a connected subset of X, for every $x \in Y$.

The following examples show that the restriction of a pointly connected mapping need not be a pointly connected mapping and the composition of pointly connected mappings need not be a pointly connected mapping.

Example 3.10. Let $f : (\mathrm{R}^+ - \mathrm{Z}^+) \to \mathrm{Z}^+ \cup \{0\}$ be a mapping defined by $f(x) = [x]$, largest integer less than or equal to x. Then f is a pointly connected mapping. Let $A = \mathrm{R}^+ - (\mathrm{Z}^+ \cup \{\frac{2n+1}{2} : n = 0, 1, 2, \cdots\cdots\})$. Then $f|_A^{-1}(m) = (m, \frac{2m+1}{2}) \cup (\frac{2m+1}{2}, m+1)$, which is not a connected subset of A.

Example 3.11. Let $f : (\mathrm{R} - \mathrm{Z}) \to \mathrm{Z}$ be a mapping defined by $f(x) = [x]$, for every $x \in (\mathrm{R} - \mathrm{Z})$and $g : \mathrm{R} - (\mathrm{Z} \cup \{\frac{2n+1}{2} : n = 0, 1, 2, \cdots\cdots\}) \to \mathrm{R} - \mathrm{Z}$ be a mapping defined by $f(x) = x$, for every $x \in \mathrm{R} - (\mathrm{Z} \cup \{\frac{2n+1}{2} : n = 0, 1, 2, \cdots\cdots\})$. Here, f and g are pointly connected mappings. But $f \circ g$ is not a pointly connected mapping, because $(f \circ g)^{-1}(m) = (m, \frac{2m+1}{2}) \cup (\frac{2m+1}{2}, m+1)$, for $m > 0$, which is not a connected subset of $\mathrm{R} - (\mathrm{Z} \cup \{\frac{2n+1}{2} : n = 0, 1, 2, \cdots\cdots\})$.

Definition 3.12. A mapping $f : X \to Y$ is said to be a connected mapping if $f^{-1}(A)$ is a connected subset of X, whenever A is a connected subset of Y.

Remark 3.13.

(1) Restriction of a connected mapping need not be a connected mapping. This can be verified from example 3.10
(2) Composition of connected mappings is a connected mapping.
(3) Every connected mapping is a pointly connected mapping. The converse need not be true.
(4) Let $f : X \to Y$ and $g : Y \to Z$ be two pointly connected mappings. Then $g \circ f$ is a pointly connected mapping if g is an one to one mapping or f is a connected mapping.
(5) Let $f : X \to Y$ be an one to one mapping from X onto Y. If f is an open mapping, then f is a connected mapping.

Definition 3.14 ([6])**.** Let $f : X \to Y$ be a mapping from X to Y and $U \subseteq X$. Then f is said to be saturated on U, if there is a subset V of Y such that $U = f^{-1}(V)$

Theorem 3.15. *Let $f : X \to Y$ be a mapping from X to Y. If f is an open mapping on every saturated subset of X and f seperates open subsets of X, then f is a connected mapping.*

Proof. Let U be an connected subset of Y. If $f^{-1}(U)$ is not connected, we can find a clopen subset V of X such that $V \cup X \backslash V = f^{-1}(U)$. Since f is an open mapping on every saturated subset of X, $f(V)$ and $f(X \backslash V)$ are open subsets of Y. Also $f(V) \cup f(X \backslash V) = U$, because f seperates open sets in X, which is a seperation on U. Thus $f^{-1}(U)$ is connected ☐

The converse of the above remark 3.13 need not be true. The following example shows it.

Example 3.16. Let X be the subspace $[0, 1] \cup (2, 3]$ of R and let Y be the subspace $[0, 2]$ of R. Define a map $f : X \to Y$ by

$f(x) = \begin{cases} x & \text{if } x \in [0,1] \\ x - 1 & \text{if } x \in (2,3] \end{cases}$. Then f is a continuous onto mapping. Also f is a connected mapping and seperates open subsets of X, but not an open mapping.

Theorem 3.17. *Let $f : X \to Y$ be a continuous and connected mapping from X onto Y. Then X is n-connected if and only if Y is n-connected.*

Proof. If f is a continuous and connected mapping, then f preserves connected subset of X onto a connected subset of Y. □

4. Remainders of Connectifications

Definition 4.1 ([8]). Let $\xi_1 X, \xi_2 X$ be two connectifications of a space X. Then, we write $\xi_1 X \geq \xi_2 X$, if there is a continuous function $f : \xi_1 X \to \xi_2 X$ such that $f(x) = x$, for all $x \in X$.

Theorem 4.2. *Let ξX be any connectification of a space X and $\{K_i : i = 1, 2, \cdots n\}$ be a collection of mutually disjoint nonempty compact subsets of $\xi X \backslash X$. Choose n distinct points $\{p_i : i = 1, 2, \cdots n\}$ not in ξX and define a mapping $h : \xi X \to \zeta X$ by $h(p) = \begin{cases} p & \text{if } p \in \xi X \backslash (\overset{n}{\underset{i=1}{\cup}} K_i) \\ p_i & \text{if } p \in K_i \end{cases}$, where $\zeta X = (\xi X \backslash \overset{n}{\underset{i=1}{\cup}} K_i) \cup \{p_1, p_2 \cdots \cdots p_n\}$. Let γX have the quotient topology induced by h. Then ζX is a connectification for X such that $\xi X \geq \zeta X$.*

Proof. Proof of this theorem is similar to the proof of Lemma: 2 in [5] □

Theorem 4.3. *Let ξX be any connectification of a regular space X. Let $\{K_i : i \in I\}$ be a collection of mutually disjoint nonempty compact subsets of $\xi X \backslash X$ and they are locally finite in ξX. Let $\{p_i : i \in I\}$ be such that $p_i \notin X$, for every $i \in I$. Then there is a connectification ζX for X such that $\xi X \geq \zeta X$.*

Proof. Let $\zeta X = (\xi X \backslash \underset{i \in I}{\cup} K_i) \cup \{p_i : i \in I\}$ and $Y = (\xi X \backslash \underset{i \in I}{\cup} K_i)$ where p_i are distinct, and $p_i \notin \xi X$. Define a map $h : \xi X \to \zeta X$ by $h(x) = \begin{cases} x & \text{if } x \in Y \\ p_i & \text{if } x \in K_i \end{cases}$. Let ζX have the quotient topology induced by the map h. Since ξX is connected, ζX is connected. Let U be an open set in ζX. Then $h^{-1}(U)$ is an open set in ξX which intersects X so that $h(h^{-1}(U)) = U$ intersects $h(X) = X$. Hence X is dense in ζX.

Consider two distinct elements y_1, y_2 in ζX. Let $A = h^{-1}(y_1)$ and $B = h^{-1}(y_2)$. For any $x \in A$, there is an open set U_x of x in ξX such that U_x intersects only finite number of K_i's and $U_x \cap B = \phi$. Define an open set $W_x = U_x \backslash (\overset{n}{\underset{i=1}{\cup}} K_i)$ in ξX containing x. Find an open set V_x of x such that $cl(V_x) \subseteq W_x$ Then $\{V_x : x \in A\}$ is an open cover of A. Since $f^{-1}(x)$ is compact, for every $x \in X$. This open cover has a finite subcover $\{V_{x_1}, V_{x_2}, \cdots V_{x_m}\}$, say. Let $U = \overset{m}{\underset{i=1}{\cup}} V_{x_i}$. Then U is an open set such that $A \subseteq U$ and $clU \cap B = \phi$. Similarly, we can find another open set V such that $B \subseteq V$ and $U \cap V = \phi$. Then $f(U)$ and $f(V)$ are disjoint open sets in ζX such that $f(A) = y_1 \in f(U)$ and $f(B) = y_2 \in f(V)$. This proves the Hausdorffness of ζX. □

Remark 4.4. In the above theorem 4.3, regularity of X is used only for Hausdorffness of ζX.

Definition 4.5 ([8]). A space X is said to be a P-space if every G_δ set is an open set in X.

Theorem 4.6. *Let X be a regular P-space having no isolated points and ξX be any connectification of X. Let $\{K_i : i \in I\}$ be a collection of mutually disjoint nonempty compact subsets of $\xi X \backslash X$ and they are locally countable in ξX. Let $\{p_i : i \in I\}$ be such that $p_i \notin X$, for every $i \in I$. Then there is a connectification ζX for X such that $\xi X \geq \zeta X$.*

Proof. Proof for connected extension is same as theorem: 4.3. Hausdorffness of connectification has some difficulties. Consider two distinct elements y_1, y_2 in ζX. Let $A = f^{-1}(y_1)$ and $B = f^{-1}(y_2)$. For any $x \in A$, find an open set U_x of x such that $U_x \cap K_i \neq \phi$, for every $i \in N_1 \subseteq$ N, the set of all natural numbers. Define an open set $W_x = U_x \backslash (\bigcup_{i=1}^{\infty} K_i)$ in ξX containing x. Find an open set V_x of x such that$clV_x \subseteq W_x$. Since $f^{-1}(x)$ is compact, for every $x \in X$, $\{V_x : x \in A\}$ has a finite subcollection $\{V_{x_1}, V_{x_2}, \cdots V_{x_m}\}$, such that $A \subseteq U = \bigcup_{i=1}^{m} V_{x_i}$. Similarly, we can find another open set V such that $B \subseteq V$ and $U \cap V = \phi$. Then $f(U)$ and $f(V)$ are disjoint open sets in ζX such that $f(A) = y_1 \in f(U)$ and $f(B) = y_2 \in f(V)$. This proves the Hausdorffness of ζX. □

5. Lattices of Connectifications

Theorem 5.1. *If Y is a dense subspace of a space X, then the lattice $\mathcal{C}(X)$ of all connectifications of X can be embedded into the lattice $\mathcal{C}(Y)$ of all connectifications of Y by an order preserving map which also preserves join.*

Proof. Since Y is dense in X, every connectification of X is also a connectification of Y. □

Theorem 5.2. *Let X and Y be two spaces having no isolated points. Let $f : X \to Y$ be a continuous and connected mapping from X onto Y such that f seperates every pair of disjoint regular open subsets of X. Then the lattice $\mathcal{C}(X)$ is isomorphic to $\mathcal{C}(Y)$.*

Proof. If ξX is any connectification of X, then by theorem 3.5, we can find a connectification ζY for Y with remainder $\xi X \backslash X$. Similarly, if ζY is any connectification of Y, then there is a connectification ξX for X with remainder $\zeta Y \backslash Y$. Thus we have a one to one correspondence from $\mathcal{C}(X)$ onto $\mathcal{C}(Y)$. Let $\xi_1 X$ and $\xi_2 X$ be two connectifications of X such that $\xi_1 X \leq \xi_2 X$, then there are two connectifications $\zeta_1 Y$ and $\zeta_2 Y$ of Y with remainders $\xi_1 X \backslash X$ and $\xi_2 X \backslash X$ such that $\xi_1 X \mapsto \zeta_1 Y$ and $\xi_2 X \mapsto \zeta_2 Y$. By our construction in theorem 3.5, we have $\zeta_1 Y \leq \zeta_2 Y$. This completes the proof of this theorem. □

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