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Concerning nearly metrizable spaces

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Abstract

The purpose of this paper is to introduce the notion of near metrizability for topological spaces, which is strictly weaker than the concept of metrizability. A number of characterizations of nearly metrizable spaces is achieved here as analogues of the corresponding ones for metrizable spaces. It is seen that near metrizability is a natural idea vis-a-vis near paracompactness, playing the similar role as played by paracompactness with regard to metrizability.

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1. Introduction and Preliminary Results

The notion of nearly paracompact space was introduced by Singal and Arya [12], and such spaces have so far been studied by many researchers with keen interest (e.g. see [3], [5], [6], [8], [10]). Now, there are well known interrelations between paracompactness and metrizability; for instance, every metrizable space is always paracompact. It is thus natural to search for the kind of spaces which take the corresponding role of metrizability vis-a-vis near paracompactness.

In this paper, we like to introduce and study a class of spaces, called nearly metrizable, which properly contains all metrizable spaces and which bears a similar kind of relationship with the class of all nearly paracompact spaces [12] as the family of all metrizable spaces has with the class of paracompact spaces. We shall show that within the class of paracompact spaces the concept

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of near metrizability and that of metrizability coincide. Thus there exist nearly metrizable, non-metrizable spaces which are not paracompact.

We recall that a subset A of a topological space X is called regular open, if $A = \text{intcl}A$ (as usual, 'int' and 'cl' stand for interior and closure operators respectively). To simplify notation, we shall write A^* instead of intcl$A$. It is easy to see that a subset of a space X is regular open iff it is of the form A^*, for some $A \subseteq X$. We denote by $RO(X)$ the family of all regular open subsets of X, i.e., $RO(X) = \{A^* : A \subseteq X\}$. It is well known [1] that $RO(X)$ is an open base for some topology τ_s on a space (X, τ). The set X endowed with this topology τ_s will be denoted by $(X)_s$ and is called the semiregularization space of X. In particular, X is called semiregular if these two topologies on X coincide. As rightly observed by Mrševic et al.[7], semiregularization topologies and the associated techniques are found quite important in the study of H-closed, minimal Hausdorff and S-closed spaces. Any subset A of X, which is open in $(X)_s$, is called δ-open [15]. A subset A of a space X is called regular closed if $X \setminus A \in RO(X)$. We denote by $RC(X)$ the family of all regular closed subsets of X. For any family $\mathcal{A}$ of subsets of X, we denote by $\mathcal{A}^\#$ the family given by $\mathcal{A}^\# = \{A^* : A \in \mathcal{A}\}$.

We now mention some simple facts which will be needed for our discussion.

Lemma 1.1. *If A and B are two open sets in X, then $A \bigcap B \neq \phi \Leftrightarrow A^* \bigcap B \neq \phi \Leftrightarrow A^* \bigcap B^* \neq \phi$.*

Lemma 1.2. *A space X is T_2 iff $(X)_s$ is T_2.*

In [11], Singal and Arya called a space X almost regular if for any $A \in RC(X)$ and any $x \in X \setminus A$, there exist disjoint open sets U and V in X such that $x \in U$ and $A \subseteq V$. The same authors in [12] called a space X nearly paracompact if every regular open cover of X has a locally finite open refinement.

We now state the following well known results (one may find them in [7]):

Theorem 1.3. *Let X be a topological space. Then*
(a) X is nearly paracompact iff $(X)_s$ is paracompact.
(b) X is almost regular iff $(X)_s$ is regular.

In the next section we introduce pseudo-embedding and thereby near metrizability. Certain characterizations of near metrizability and its study vis-a-vis paracompactness and near paracompactness are taken up in this section. In Section 3, two other notions viz. pseudo-bases and local pseudo-bases are defined to facilitate further investigations of near metrizability, where we will show, among other things, that a space X is nearly metrizable iff $(X)_s$ is metrizable. The last section consists of just two characterizations of near metrizability deduced from perspectives different from those in the earlier sections.

2. Pseudo-embedding and Near Metrizability

We begin by introducing the idea of pseudo-embedding as a generalized concept of an embedding.

Definition 2.1. *If X and Y are two topological spaces, then a continuous, injective map $f : X \to Y$ is called a pseudo-embedding of X into Y, if for any $A \in RO(X)$, $f(A)$ is open. If there is a pseudo-embedding f of X into Y, then we say that X is pseudo-embeddable in Y. If a pseudo-embedding $f : X \to Y$ is surjective, we say that f is a pseudo-embedding of X onto Y.*

It is easy to see that every embedding is a pseudo-embedding; but the converse is false as is shown in the following example.

Example 2.2. Let $\mathbb{R}$ be the set of all real numbers; and, τ_f and τ_c be respectively the co-finite topology and the co-countable topology on $\mathbb{R}$. Since $\tau_f \subseteq \tau_c$, the identity map $i : (\mathbb{R}, \tau_c) \to (\mathbb{R}, \tau_f)$ is a continuous bijection which maps every regular open subsets of $(\mathbb{R}, \tau_c)$ onto an open subset of $(\mathbb{R}, \tau_f)$ (note that $\mathbb{R}$ and ϕ are the only regular open sets in $(\mathbb{R}, \tau_c)$). Hence i is a pseudo-embedding; but it is not an embedding, because $\mathbb{R} \setminus \mathbb{Q} \in \tau_c$ whereas $\mathbb{R} \setminus \mathbb{Q} \notin \tau_f$ ($\mathbb{Q}$ denoting the set of all rational numbers).

Remark 2.3. If X is semiregular, then $RO(X)$ makes an open base for the topology of X and hence any pseudo-embedding of a semi-regular space X into any space Y is an embedding.

Definition 2.4. *A space X is called nearly metrizable if it is pseudo-embeddable in a metric space Y.*

Remark 2.5. It is obvious that every metrizable space is nearly metrizable; we show below that the converse does not hold, in general.

Example 2.6. Let τ_1 and τ_2 respectively denote the Euclidean and co-countable topologies on the set $\mathbb{R}$ of all real numbers, and let τ be the smallest topology on $\mathbb{R}$ generated by $\tau_1 \bigcup \tau_2$. Since $\tau_1 \subseteq \tau$, the identity map $i : (\mathbb{R}, \tau) \to (\mathbb{R}, \tau_1)$ is a continuous bijection. Since the regular open subsets of $(\mathbb{R}, \tau)$ are precisely the sets which are regular open in $(\mathbb{R}, \tau_1)$ (see Example 63 of Steen and Seeback [14]), i maps regular open subsets of $(\mathbb{R}, \tau)$ onto open subsets of $(\mathbb{R}, \tau_1)$ and hence i becomes a pseudo-embedding. Since $(\mathbb{R}, \tau_1)$ is a metrizable space, it follows that $(\mathbb{R}, \tau)$ is nearly metrizable; but $(\mathbb{R}, \tau)$ is not metrizable as it is not regular.

Remark 2.7. (a) In view of Remark 2.3 it follows that a semiregular, nearly metrizable space is metrizable.
(b) Since metrizability is a hereditary property, it follows that a space X is nearly metrizable iff there is a pseudo-embedding f from X onto a metrizable space Y.

We now prove a few properties of nearly metrizable spaces.

Theorem 2.8. *Every nearly metrizable space is Hausdorff and almost regular.*

Proof. We only prove that a nearly metrizable space X is almost regular, the Hausdorffness of X can similarly be proved.
By near metrizability of X, there is a pseudo-embedding f of X onto a metric space Y. Let $A \in RC(X)$ and $x \in X \setminus A$. Since $f : X \to Y$ is a bijection and f maps regular open subsets of X onto open subsets of Y, it follows that $f(A)$ is closed in Y and $f(x) \in Y \setminus f(A)$. Since Y is regular, there exist disjoint open sets U and V in Y such that $f(x) \in U$ and $f(A) \subseteq V$. It is now easy to see that $f^{-1}(U)$ and $f^{-1}(V)$ are two disjoint open sets in X with $x \in f^{-1}(U)$ and $A \subseteq f^{-1}(V)$. Thus X is almost regular. □

Remark 2.9. We shall call an almost regular, Hausdorff space an almost T_3-space. So, what we have proved in the above theorem is that a nearly metrizable space is an almost T_3-space.

It is well known that every metrizable space is paracompact. But we give here an example to show that a nearly metrizable space may not be paracompact.

Example 2.10. Consider the nearly metrizable space $(\mathbb{R}, \tau)$ of Example 2.6. Since every Hausdorff, paracompact space is regular and $(\mathbb{R}, \tau)$ is not regular, it cannot be paracompact.

It is then a natural **question**: When does a nearly metrizable space become paracompact ? The following result answers it.

Theorem 2.11. *A nearly metrizable space X is paracompact iff it is metrizable.*

Proof. For the necessity, let X be a nearly metrizable, paracompact space. Since X is Hausdorff (by Theorem 2.8) and paracompact, it is regular. As a semiregular, nearly metrizable space is metrizable (see Remark 2.7), X becomes metrizable.
The sufficiency part is clear. □

It then follows from Example 2.10 and the above theorem that a nearly metrizable and non-metrizable space is never paracompact. However, as expected, we show below that every nearly metrizable space is nearly paracompact.

Theorem 2.12. *Every nearly metrizable space X is nearly paracompact.*

Proof. Let X be a nearly metrizable space and $\mathcal{U} \subseteq RO(X)$ be a cover of X. Then there is a pseudo-embedding f of X onto a metrizable space Y. Since f maps regular open subsets of X onto open subsets of Y, it follows that $\{f(A) : A \in \mathcal{U}\}$ is an open cover of Y. Since Y is paracompact (being a metric space), there is an open locally finite refinement $\mathcal{V}$ of $\{f(A) : A \in \mathcal{U}\}$ in Y. It is then easy to check that $\{f^{-1}(V) : V \in \mathcal{V}\}$ is a locally finite open refinement of $\mathcal{U}$ in X and hence X is nearly paracompact. □

We conclude this section by giving a sufficient condition for near metrizability.

Theorem 2.13. *If the semiregularization space $(X)_s$ of a space X is metrizable, then X is nearly metrizable.*

Proof. If $(X)_s$ is metrizable, then as the topology of X is finer than that of $(X)_s$, the identity map $i : X \to (X)_s$ is a pseudo-embedding of X onto $(X)_s$. Hence X is nearly metrizable. □

3. (Local) Pseudo-bases and Near Metrizability

In this section we shall give some characterizations of nearly metrizable spaces by introducing the ideas of pseudo-bases and local pseudo-bases. We shall, in addition, prove the converse of Theorem 2.13.

Definition 3.1. *Suppose $\mathcal{B}$ is a family of open subsets of X. We say that $\mathcal{B}$ is a pseudo-base in X if for any $A \in RO(X)$, there is a subfamily $\mathcal{B}_0$ of $\mathcal{B}$ such that $A = \bigcup\{B : B \in \mathcal{B}_0\}$.*

We call a pseudo-base $\mathcal{B}$ σ-locally finite if $\mathcal{B}$ can be expressed as $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$, where $\mathcal{B}_n$ is locally finite, for each $n \in \mathbb{N}$.

It is obvious that every base is a pseudo-base; but the converse is false as is shown in the following example.

Example 3.2. Let $\mathbb{R}$ be the set of reals and τ be the co-countable topology on $\mathbb{R}$. Then $\mathcal{B} = \{\mathbb{R}, \phi\}$ is a pseudo-base for $(\mathbb{R}, \tau)$, but is not a base for it.

Remark 3.3. For a semiregular space X, $RO(X)$ makes an open base for X so that every pseudo-base in a semiregular space is a base.

We now prove a lemma which will be very useful for the rest of the paper.

Lemma 3.4. *Suppose $\mathcal{B}$ is a family of open subsets of a space X. If $\mathcal{B}$ is a pseudo-base in X then $\mathcal{B}^{\#} = \{B^* : B \in \mathcal{B}\}$ is a base for the topology of $(X)_s$.*

Proof. It is enough to show that each member of $RO(X)$ can be expressed as a union of some members of $\mathcal{B}^{\#}$, as $RO(X)$ is an open base for the topology of $(X)_s$. So, let $A \in RO(X)$. Since $\mathcal{B}$ is a pseudo-base in X, there exists a subfamily $\mathcal{B}_0$ of $\mathcal{B}$ such that $A = \bigcup\{B : B \in \mathcal{B}_0\}$. Therefore, $B \subseteq A$, for all $B \in \mathcal{B}_0$, i.e., $B^* \subseteq A$, for all $B \in \mathcal{B}_0$ (since $A \in RO(X)$, $A^* = A$). Also, $B \subseteq B^*$, for all $B \in \mathcal{B}_0$. Thus $B \subseteq B^* \subseteq A$, for all $B \in \mathcal{B}_0$, i.e., $\bigcup\{B : B \in \mathcal{B}_0\} \subseteq \bigcup\{B^* : B \in \mathcal{B}_0\} \subseteq A$, i.e., $A = \bigcup\{B^* : B \in \mathcal{B}_0\}$ and hence $\mathcal{B}^{\#}$ becomes a base for the topology of $(X)_s$. □

Remark 3.5. In the above lemma, if $\mathcal{B}^{\#}$ is a base for $(X)_s$, then clearly $\mathcal{B}^{\#}$ is a pseudo-base of X, but $\mathcal{B}$ is not necessarily a pseudo-base for X. In fact, for the space $\mathbb{R}$ with co-countable topology τ we have $\tau_s = \{\phi, \mathbb{R}\}$. If we take $\mathcal{B} = \{\mathbb{R} \setminus \{1\}, \mathbb{R} \setminus \{1, 2\}, ...\}$ then clearly $\mathcal{B}^{\#} = \{\mathbb{R}, \phi\}$ is a base for $(\mathbb{R})_s$, but $\mathcal{B}$ is not a pseudo-base in $\mathbb{R}$.

The first characterization which we now give for nearly metrizable spaces is quite similar to the celebrated Nagata-Smirnov metrization theorem which states that 'A T_3-space X is metrizable iff it has a σ-locally finite open base' (see [9]). One part of the proposed result goes as follows.

Theorem 3.6. *An almost T_3-space X possessing a σ-locally finite pseudo-base is nearly metrizable.*

Proof. Let $\mathcal{B}$ be a σ-locally finite pseudo-base in X. Then $\mathcal{B}$ can be expressed as $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$, where each $\mathcal{B}_n$ is locally finite in X. We claim that $\mathcal{B}^{\#} = \bigcup_{n=1}^{\infty} \mathcal{B}_n^{\#}$ is a σ-locally finite base for $(X)_s$. That $\mathcal{B}^{\#}$ is a base for $(X)_s$ follows from Lemma 3.4. We now show that $\mathcal{B}^{\#}$ is a σ-locally finite family i.e., we show that $\mathcal{B}_n^{\#}$ is a locally finite family in $(X)_s$ for each $n \in \mathbb{N}$. For this, let $x \in X$. Since for each $n \in \mathbb{N}$, $\mathcal{B}_n$ is locally finite in X, there exists an open neighbourhood U of x in X which intersects at most finitely many members of $\mathcal{B}_n$ for each $n \in \mathbb{N}$. Now, U^* is an open neighbourhood of x in $(X)_s$ and, in view of Lemma 1.1, U^* can intersect at most finitely many members of $\mathcal{B}_n^{\#}$, for each $n \in \mathbb{N}$. Thus $\mathcal{B}^{\#}$ is a σ-locally finite base for $(X)_s$. Since X is almost T_3, by Lemma 1.2 and Theorem 1.3(b), $(X)_s$ is a T_3-space. Thus by Nagata-Smirnov metrization theorem, $(X)_s$ is metrizable and hence, in view of Theorem 2.13, X is nearly metrizable. ☐

Corollary 3.7. *An almost T_3-space with a countable pseudo-base is nearly metrizable.*

Proof. Follows directly from Theorem 3.6, since every countable family is a σ-locally finite family. ☐

Theorem 3.8. *Every nearly metrizable space admits a σ-locally finite pseudo-base.*

Proof. If X is nearly metrizable then there is a pseudo-embedding f of X onto a metric space Y. By Nagata-Smirnov metrization theorem, there is a σ-locally finite open base $\mathcal{B}$ in Y. Let $\mathcal{A} = \{f^{-1}(B) : B \in \mathcal{B}\}$. Then $\mathcal{A}$ is a family of open sets in X. We show that $\mathcal{A}$ is a pseudo-base in X. For that, let $A \in RO(X)$. Since $f(A)$ is open in Y, there is a subfamily $\mathcal{B}_0$ of $\mathcal{B}$ such that $f(A) = \bigcup\{B : B \in \mathcal{B}_0\}$ which, in turn, implies that $A = \bigcup\{f^{-1}(B) : B \in \mathcal{B}_0\}$. Thus $\mathcal{A}$ becomes a pseudo-base in X. That $\mathcal{A}$ is σ-locally finite is clear and hence the result follows. ☐

As a consequence of Theorem 2.8, 3.6 and 3.8 we have :

Theorem 3.9. *A space X is nearly metrizable iff it is almost T_3 and possesses a σ-locally finite pseudo-base.*

We now prove the converse of Theorem 2.13.

Theorem 3.10. *If a space X is nearly metrizable, then $(X)_s$ is metrizable.*

Proof. If X is nearly metrizable, then by Theorem 2.8 and 3.8 it follows that X is almost T_3 and has a σ-locally finite pseudo-base $\mathcal{B}$. Then $\mathcal{B}^{\#}$ is a σ-locally finite open base for $(X)_s$ (see the proof of Theorem 3.6). Since X is almost T_3, by Lemma 1.2 and Theorem 1.3(b), $(X)_s$ is T_3 and hence by Nagata-Smirnov metrization theorem, $(X)_s$ is metrizable. □

Question: Does there exist any direct proof of the above theorem without using Nagata-Smirnov metrization theorem?

Combining Theorems 2.13 and 3.10, we obtain :

Theorem 3.11. *A space X is nearly metrizable iff $(X)_s$ is metrizable.*

Remark 3.12. As observed in Corollary 1 of [7] , the half-disc topology (Counterexample 78 of [14]) gives an example of a space X which is not regular, but such that $(X)_s$ is metrizable. Then by Theorem 3.11 it follows that X is another example of a space which is nearly metrizable but is not metrizable.

We now introduce the concept of local pseudo-bases which will also be used to obtain some further characterizations of near metrizability.

Definition 3.13. *Suppose X is a topological space and $x \in X$. A family $\mathcal{A}$ of open subsets of X each of which contains x, is called a local pseudo-base at x, if for any $B \in RO(X)$ with $x \in B$, there exists an $A \in \mathcal{A}$ such that $A \subseteq B$.*

It is obvious that any local base at x in a space X is a local pseudo-base at x; but in the following example we show that the converse may not be true.

Example 3.14. Let $(\mathbb{R}, \tau)$ be the space of real numbers endowed with the co-countable topology τ. Then $\mathcal{A} = \{\mathbb{R}, \phi\}$ is a local pseudo-base at x for each $x \in \mathbb{R}$; but $\mathcal{A}$ is not a local base at any point of $\mathbb{R}$.

We shall use the following result for our discussion in the sequel.

Lemma 3.15. *Let $\mathcal{A}$ be a family of open subsets of X each of which contains $x(\in X)$. Then $\mathcal{A}$ is a local pseudo-base at x in X iff $\mathcal{A}^{\#}$ is a local base at x in $(X)_s$.*

Proof. Obviously, $\mathcal{A}^{\#}$ is a family of open subsets of $(X)_s$ each of which contains x. Let $B \in RO(X)$ and $x \in B$. Since $\mathcal{A}$ is a local pseudo-base at x in X, there exists some $A \in \mathcal{A}$ such that $A \subseteq B$ which gives $A^* \subseteq B$ (as $B \in RO(X), B^* = B$), where $A^* \in \mathcal{A}^{\#}$ and hence $\mathcal{A}^{\#}$ becomes a local base at x in $(X)_s$ (since $RO(X)$ is an open base for the topology of $(X)_s$).
Conversely, let $x \in B \in RO(X)$. As $\mathcal{A}^{\#}$ is a local base at x in $(X)_s$, there exists some $A \in \mathcal{A}$ such that $x \in A \subseteq B$. Then $x \in A \subseteq \text{int}(cl(A)) \subseteq \text{int}(cl(B)) = B$. Thus $\mathcal{A}$ is a local pseudo-base at x in X. □

The following is a well known characterization for metrizable spaces, which may be found in page 192 of [9].

Theorem 3.16. *A T_1- space X is metrizable iff there exists a countable local base $\{B_n(x) : n \in \mathbb{N}\}$ at x, for each $x \in X$ such that for every $n \in \mathbb{N}$, there exists an $m \in \mathbb{N}$ for which $B_m(x) \bigcap B_m(y) \neq \phi$ implies that $B_m(y) \subseteq B_n(x)$, for any $y \in X$.*

We now prove an analogous version of the above result for nearly metrizable spaces:

Theorem 3.17. *A T_2-space X is nearly metrizable iff there exists a countable local pseudo-base $\{B_n(x) : n \in \mathbb{N}\}$ at x, for each $x \in X$ such that for every $x \in X$ and every $n \in \mathbb{N}$, there exists an $m \in \mathbb{N}$ for which $B_m(x) \bigcap B_m(y) \neq \phi$ implies that $B_m(y) \subseteq B_n(x)$, for any $y \in X$.*

Proof. Since X is T_2, $(X)_s$ is T_1 (in fact, T_2).
First, let X be nearly metrizable. Then by Theorem 3.10, $(X)_s$ is metrizable. Thus by Theorem 3.16, there exists a countable local base $\mathcal{B}_x$ at each x in $(X)_s$ satisfying the condition of the hypothesis and hence the necessity follows.
Conversely, let $\mathcal{B}_x = \{B_n(x) : n \in \mathbb{N}\}$ be a local pseudo-base at x, for each $x \in X$ such that the given condition holds. By Lemma 3.15, $\mathcal{B}_x^{\#}$ is a countable local base at x in $(X)_s$, for each $x \in X$. We now show that these $\mathcal{B}_x^{\#}$'s satisfy the hypothesis of Theorem 3.16. For that, let $x \in X$ and $n \in \mathbb{N}$. Then there exists $m \in \mathbb{N}$ such that ($B_m(x) \bigcap B_m(y) \neq \phi \Rightarrow B_m(y) \subseteq B_n(x)$, for each $y \in X$). Now $B^*_m(x) \bigcap B^*_m(y) \neq \phi \Rightarrow B_m(x) \bigcap B_m(y) \neq \phi \Rightarrow B_m(y) \subseteq B_n(x)$ $\Rightarrow B^*_m(y) \subseteq B^*_n(x)$ for each y in X. Then by Theorem 3.16, $(X)_s$ is metrizable and hence, in view of Theorem 2.13, X is nearly metrizable. □

Another nice characterization of metrizable spaces can be found in [9]. It goes as follows:

Theorem 3.18. *A T_1-space X is metrizable iff for each $x \in X$, there exist two sequences $\{A_n(x) : n \in \mathbb{N}\}$ and $\{B_n(x) : n \in \mathbb{N}\}$ of open neighbourhoods of x in X such that*
(i) $\{A_n(x) : n \in \mathbb{N}\}$ is a local base at x, for each $x \in X$,
(ii) $y \notin A_n(x) \Rightarrow B_n(x) \bigcap B_n(y) = \phi$, and
(iii) $y \in B_n(x) \Rightarrow B_n(y) \subseteq A_n(x)$.

Remark 3.19. It is easy to see that in the above theorem we can assume that $B_n(x) \in \mathcal{B}$ (for all $n \in \mathbb{N}$) for a given open base $\mathcal{B}$ for X.

An analogue of the above theorem for near metrizability is now proved.

Theorem 3.20. *A T_2-space X is nearly metrizable iff for each $x \in X$, there exist two sequences $\{A_n(x) : n \in \mathbb{N}\}$ and $\{B_n(x) : n \in \mathbb{N}\}$ of open neighbourhoods of x in X with $B_n(x) \in RO(X)$, for all $n \in \mathbb{N}$ satisfying the following conditions:*
(i) $\{A_n(x) : n \in \mathbb{N}\}$ is a local pseudo-base at x, for each $x \in X$,
(ii) $y \notin A_n(x) \Rightarrow B_n(x) \bigcap B_n(y) = \phi$, and
(iii) $y \in B_n(x) \Rightarrow B_n(y) \subseteq A_n(x)$.

Proof. First let X be nearly metrizable. Then $(X)_s$ is metrizable and also T_2 (as X is T_2). Thus by Theorem 3.18, there exist two sequences $\{A_n(x) : n \in \mathbb{N}\}$ and $\{B_n(x) : n \in \mathbb{N}\}$ of open neighbourhoods of x in $(X)_s$ (and hence in X) satisfying the conditions of the above theorem, where, in view of Remark 3.19, we can assume that $B_n(x) \in RO(X)$ for all $n \in \mathbb{N}$ as $RO(X)$ is an open base for $(X)_s$.
Conversely, let for each $x \in X$, there exist two sequences $\{A_n(x) : n \in \mathbb{N}\}$ and $\{B_n(x) : n \in \mathbb{N}\}$ of open neighbourhoods of x in X with $B_n(x) \in RO(X)$, for all $n \in \mathbb{N}$ such that
(i) $\{A_n(x) : n \in \mathbb{N}\}$ is a local pseudo-base at x, for each $x \in X$;
(ii) $y \notin A_n(x) \Rightarrow B_n(x) \bigcap B_n(y) = \phi$, and
(iii) $y \in B_n(x) \Rightarrow B_n(y) \subseteq A_n(x)$.
Then $\{A_n^*(x) : n \in \mathbb{N}\}$ and $\{B_n(x) : n \in \mathbb{N}\}$ are two sequences of open neighbourhoods of x in $(X)_s$ such that
(a) $\{A_n^*(x) : n \in \mathbb{N}\}$ is a local base at x in $(X)_s$, for each $x \in X$ (see Lemma 3.15),
(b) $y \notin A_n^*(x) \Rightarrow y \notin A_n(x)$ (since $A_n(x) \subseteq A_n^*(x)$) $\Rightarrow B_n(x) \bigcap B_n(y) = \phi$, and
(c) $y \in B_n(x) \Rightarrow B_n(y) \subseteq A_n(x) \Rightarrow B_n(y) \subseteq A_n^*(x)$ (since $A_n(x) \subseteq A_n^*(x)$).
This shows that $\{A_n^*(x) : n \in \mathbb{N}\}$ and $\{B_n(x) : n \in \mathbb{N}\}$ satisfy all the conditions of the hypothesis of Theorem 3.18 for $(X)_s$ and hence $(X)_s$ becomes metrizable which, in turn, implies that X is nearly metrizable. □

4. Two More Characterizations of Nearly Metrizable Spaces

This section is meant for deriving two more characterizations of nearly metrizable spaces from two different perspectives, which are similar versions of two well known characterizations of metrizable spaces. The first such latter characterization, which may be found in [9], goes as follows:

Theorem 4.1. *A compact, T_2-space X is metrizable iff the diagonal $\Delta_X = \{(x, x) : x \in X\}$ is G_δ-set in $X \times X$.*

We need a few definitions to arrive at an analogous version of the above theorem.

Definition 4.2 ([13])**.** *A space X is called nearly compact if every regular open cover of X has a finite subcover.*

Remark 4.3. Since $RO(X)$ is an open base for $(X)_s$, it follows that X is nearly compact iff $(X)_s$ is compact.

Definition 4.4. *A subset B of a space X will be called a regular G_δ-set in X if there is a sequence $\{B_n : n \in \mathbb{N}\}$ of δ-open sets such that $B = \bigcap_{n=1}^{\infty} B_n$.*

Obviously, every regular G_δ-set in X is a G_δ-set in X; but the converse fails as we see below:

Example 4.5. Let $X = (\mathbb{R}, \tau)$, where $\mathbb{R}$ is the set of all real numbers and τ is the co-countable topology on $\mathbb{R}$. Then $(\mathbb{R} \setminus \mathbb{Q})$ is a G_δ-set in X but is not regular G_δ.

Remark 4.6. It is easy to see that a subset B of a space X is a regular G_δ-set in X iff it is a G_δ-set in $(X)_s$.

Theorem 4.7. *A nearly compact, T_2 space X is nearly metrizable iff the diagonal $\Delta_X = \{(x, x) : x \in X\}$ is a regular G_δ-set in $X \times X$.*

Proof. If X is nearly compact and T_2, then $(X)_s$ is compact and T_2. Thus X is nearly metrizable $\Leftrightarrow$ $(X)_s$ is metrizable (by Theorem 3.11) $\Leftrightarrow$ the diagonal Δ_X is a G_δ-set in $(X)_s \times (X)_s$ (by Theorem 4.1) $\Leftrightarrow$ the diagonal Δ_X is a G_δ-set in $(X \times X)_s$ (as $(X)_s \times (X)_s = (X \times X)_s$ [4]) $\Leftrightarrow$ the diagonal Δ_X is a regular G_δ-set in $X \times X$ (by Remark 4.6). □

Definition 4.8 ([2]). *Suppose $\mathcal{A}$ is a cover of X by means of subsets of X. A sequence $\{A_n : n \in \mathbb{N}\}$ of open covers of X is called locally starring for $\mathcal{A}$ if for any $x \in X$, there are an open neighbourhood U of x in X and an $n \in \mathbb{N}$ such that $St(U, \mathcal{A}_n) \subseteq A$, for some $A \in \mathcal{A}$ (where $St(U, \mathcal{A}_n) = \bigcup\{A \in \mathcal{A}_n : A \bigcap U \neq \phi\}$).*

The following characterization of metrizability is due to Arhangelskii (see [2]).

Theorem 4.9. *A T_1-space X is metrizable iff there is a sequence $\{A_n : n \in \mathbb{N}\}$ of open covers of X that is locally starring for every open cover of X.*

Our last characterization of near metrizability in this paper is an offshoot of the above theorem.

Theorem 4.10. *A T_2 space (X, τ) is nearly metrizable iff there is a sequence $\{\mathcal{A}_n : n \in \mathbb{N}\}$ of τ-open covers of X that is locally starring in (X, τ) for every regular open cover of X.*

Proof. Since X is T_2, $(X)_s$ is T_2.
If X is nearly metrizable, then $(X)_s$ is metrizable and hence by Theorem 4.9, there is a sequence $\{\mathcal{A}_n : n \in \mathbb{N}\}$ of τ_s-open covers of X that is locally starring in $(X)_s$ for every regular open cover of X. Then clearly $\{\mathcal{A}_n : n \in \mathbb{N}\}$ is a sequence of τ-open cover of X that is locally starring in X for every regular open cover of X.
Conversely, let $\{\mathcal{A}_n : n \in \mathbb{N}\}$ be a sequence of τ-open covers of X, that is locally starring in (X, τ) for every regular open cover of X. Then $\{\mathcal{A}_n^{\#} : n \in \mathbb{N}\}$ is a sequence of τ_s-open covers of X with $\mathcal{A}_n^{\#} \subseteq RO(X)$, for all $n \in \mathbb{N}$, which can be checked (by use of Lemma 1.1) to be also a locally starring in $(X)_s$ for every regular open cover of X. Since $RO(X)$ is an open base for $(X)_s$, in view of Theorem 4.9, $(X)_s$ becomes metrizable so that X is nearly metrizable. □

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