A delayed nonlinear stochastic model for cocaine consumption: Stability analysis and simulation using real data

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abstract: In this paper we propose a stochastic mathematical model with distributed delay in order to describe the transmission dynamics of cocaine consumption in Spain. We investigate conditions to guarantee the stability in probability of the equilibrium points under stochastic perturbations via the white noise processes. The results are applied to the model cocaine consumption using data retrieved from the Spanish Drug National Plan, http://www.pnsd.mscbs.gob.es/. The obtained results may be useful for policy health authorities in order to improve the strategies against the drug consumption in the long-run.

1 Introduction

Drug consumption is a serious public health concern. In Spain it is increasing over the last years, [16]. To deal with this problem, the Spanish Health Ministry developed a Drug National Plan (DNP) with two main objectives: the first one focused on preventing drug consumption, bringing awareness of different related diseases, delaying the age of the first contact with drugs, etc; the second on implementing new treatments, evaluating current therapy programmes, trying to increase professional competence of people who work with drug consumers, etc. In order to reflect the impact of the DNP, every two years, the Spanish Health Ministry publishes surveys collecting the percentage of the misuse of different drugs, including alcohol, tobacco, cocaine, etc., so that the evolution of the prevalence of drugs consumption can be assessed.

The goal of this paper is to model the transmision dynamics of cocaine consumption using an epidemiological modelling approach. In agreement with [3], individual habits are shaped by the influence of our peers. Concretely, in this paper, cocaine consumption is going to be considered as a social habit that may be transmitted by influence of people in our environment (peers' influence in our social network). On the other hand, the study of the long-run trend of cocaine consumption may be useful for health policy makers in order to assess the effectiveness of the current policies. As we will see later the stability analysis of the proposed mathematical model is one of the main objectives of the present paper.

The transmission dynamics of social habits have been study using different approaches. In [6] a system of differential equations is proposed to study the tobbaco smoke dynamics as well as how the new laws in rule have affect it. An homotopy technique to numerically solve this tobbaco model is presented in [7]. In [12], a deterministic system of differential equations to study the dynamics of the alcohol consumption in Spain over the time is proposed. In [14, Chapter 12] stability of the above model is studied under stochastic perturbations of the white noise type. Recently, in [1] the authors have proposed a dynamical model to describe the use of the electronic commerce in Spain. This study includes a full stochastic stability analysis of equilibrium points under stochastic perturbations.

Regarding the cocaine consumption modelling, in [2] the authors formulate a dynamic mathematical model to model the relation between the personality and drug consumption. In [11] the authors present a deterministic system of differential equations to describe the transmission dynamics of cocaine consumption in Spain. Furthermore, in [13] the same authors propose a social network model to study its short term evolution. In [8] a homothopy approach is introduced to numerically solve the system of differential equations proposed in [11]. The aim of this paper is to introduce some additional valuable aspects in modelling the transmision of cocaine consumption that were not contemplated in the model formulation proposed in [11].

It is remarkable that our decisions and habits are directly influenced by our peers [3]. This fact applies to the cocaine abuse. However, the cocaine consumption does not start immediately after the encounters with cocaine abusers, and the transition to become a consumer requires certain time lag. Moreover, there are also independent and complex factors in the transmission of cocaine consumption, as behaviour, personality, habits, etc., whose nature is not deterministic because it contains a degree of uncertainty. Also it is known that the avaliable data from [16] is retrieved from surveys and it comprises sampling errors. These reasons aim us to model the dynamics of cocaine consumption in Spain considering both delay and randomness in the mathematical formulation. As the long-run behaviour of cocaine consumption may be affected by uncertainty factors, we perform an analysis of stability under stochastic perturbations around the equilibrium points. This paper is organised as follows. Section 2 is devoted to build the deterministic model of transmission dynamics of cocaine consumption including the delay in the transmission term. Furthermore, in that section the model is calibrated and model parameter values are obtained to portray the Spanish data. In Section 3 the equilibrium points of the deterministic model are calculated. The randomness of the model is introduced, via stochastic perturbations around the equilibrium points, in Section 4. In Section 5, we give sufficient conditions to guarantee the steady state of the delayed stochastic model is stable in the probability sense. Section 6 is devoted to carry out numerical simulations by the retarded stochastic model in order to illustrate the application of our theoretical results and to construct predictions of cocaine consumption in Spain in the long-run. Finally, conclusions are outlined in Section 7.

2 A dynamic mathematical model to study cocaine consumption with delay

This section is addressed to introduce the mathematical model proposed in this paper to study the dynamics of cocaine consumption in Spain over the time. This model is inspired in a previous deterministic mathematical model by one of the coauthors that was presented in [11]. For the sake of clarity, later we will raise the differences between both models.

2.1 Data for Spanish cocaine consumption

We are going to work with data in Table 1. This table collects the percentage of Spanish people who consumed cocaine during the period $2001-2017$. These data has been retrieved from the Spanish Health Ministry Report [16, p. 96].

Percentages	Dec 2001	Dec 2003	Dec 2005	Dec 2007	Dec 2009
Non-consumers	91.4%	90.3%	88.4%	87.4%	86.0%
Occasional consumers	4.8%	5.9%	7.0%	8.0%	10.2%
Regular consumers	2.5%	2.7%	3.0%	3.0%	2.6%
Habitual consumers	1.3%	1.1%	1.6%	1.6%	1.2%
Percentages	Dec 2011	Dec 2013	Dec 2015	Dec 2017	
Non-consumers	87.9%	86.7%	88.3%	86.9%	
Occasional consumers	8.8%	10.2%	8.9%	10.0%	
Regular consumers	2.2%	2.1%	1.9%	2.0%	
Habitual consumers	1.1%	1.0%	0.9%	1.1%	

Table 1: Percentage of non-consumers, occasional consumers, regular consumers and habitual consumers of cocaine during the period $2001-2017$ for Spanish population aged $15-64$, [16].

2.2 Mathematical model formulation

Following the approach proposed in [11], we are going to consider the cocaine consumption as an addiction that spreads through social peer pressure or social contact. These social contacts have an influence on the transmission rate of cocaine consumption.

To conduct our study, according to the Spanish Health Ministry [16], we are going to consider the Spanish population between $15-64$ years old. This population is divided into the four groups shown in Table 1 and defined as (t in months):

- $N(t)$: Non-Consumers, percentage of population who has never consumed cocaine at time t.
- $C_o(t)$: Occasional consumers, percentage of population who has consumed sometimes cocaine in their life at time t.

- $C_r(t)$: Regular consumers, percentage of population who has consumed cocaine in the last year at time t.
- $C_b(t)$: Habitual consumers, percentage of population who has consumed cocaine in the last month at time t.

The total population is defined as $P(t) = N(t) + C_o(t) + C_r(t) + C_b(t)$, for each $t \geq 0$.

As in [11], we assume homogenous population mixing [10]; the transition of individuals between each subpopulations is determined as follows:

- We assume that the new 15-years-old individuals who enter in the system have never consumed cocaine before, that is, they will be in $N(t)$ subpopulation. It is modeled by $\mu P(t)$, where $\mu > 0$ is the monthly birth rate in Spain, since the death rate of people $0 - 15$ years old is negligible, [17].
- An individual, that belongs to subpopulation $N(t)$, starts consuming cocaine by peer pressure (influence) of cocaine consumers, $C_o(t), C_r(t), C_b(t)$, and moves to $C_o(t)$ at rate $\beta > 0$, and it is modelled by the nonlinear term $\beta N(t)(C_o(t) + C_r(t) + C_b(t))$.
- Once an individual of subpopulation $C_o(t)$ begins to consume cocaine he/she may become a regular consumer, $C_r(t)$, at rate $\gamma > 0$, and this transition is modelled by the linear term $\gamma C_o(t)$.
- If a person in $C_r(t)$ increases his/her cocaine consumption, he/she may become a habitual consumer, $C_b(t)$, at rate $\sigma > 0$, and it is modelled by $\sigma C_r(t)$.
- An individual in $C_b(t)$ may move to $N(t)$ subpopulation if he/she decides to give up cocaine consumption, goes into therapy and he/she does not consume cocaine in at least 6 months. It is modelled by the linear term $\varepsilon C_b(t)$, where $\varepsilon > 0$ is the transition rate.

As it has been pointed out in Section 1, in this study we introduce relevant mathematical differences with respect to the deterministic model described in [11], namely, randomness and a delay which makes the mathematical model more realistic. This leads us to formulate a more complex model which treatment requires advanced technical mathematical tools. With the spirit of not to complicate too much the subsequent analysis, in the present study, we assume that the death rate d is the same for all the subpopulations and equal to the birth rate, μ, that is, $d = \mu$. As a consequence, the total population is constant, i.e., $P(t) = 1$ for all $t \geq 0$.

Using the above assumptions, a dynamic cocaine consumption model for Spanish population is given by the following system of non-linear ordinary dif-

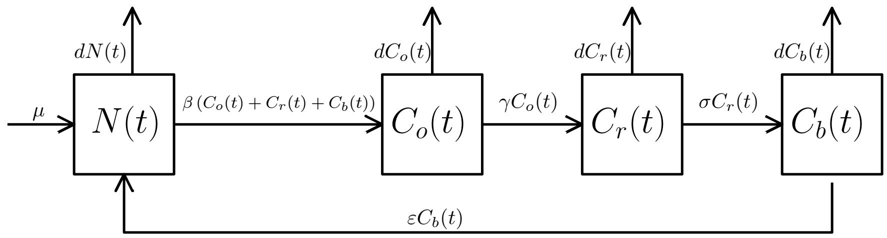

Figure 1: Compartmental diagram of the dynamic model for cocaine consumption depicted from equations (1). The boxes represent the four different subpopulations and the arrows the transitions between them.

ferential equations:

$$\begin{aligned}
\dot{N}(t) &= \mu - dN(t) - \beta N(t)(C_o(t) + C_r(t) + C_b(t)) + \varepsilon C_b(t), \\
\dot{C}_o(t) &= \beta N(t)(C_o(t) + C_r(t) + C_b(t)) - dC_o(t) - \gamma C_o(t), \\
\dot{C}_r(t) &= \gamma C_o(t) - dC_r(t) - \sigma C_r(t), \\
\dot{C}_b(t) &= \sigma C_r(t) - dC_b(t) - \varepsilon C_b(t).
\end{aligned} \tag{1}$$

A compartmental diagram representing model (1) is shown in Figure 1. The boxes represent the four subpopulations described above and the arrows represent the transitions between them.

As the total population is constant and equal to 1, then, $C_b(t) = 1 - N(t) - C_o(t) - C_r(t)$, and the system (1) can be simplified as follows:

$$\begin{aligned}
\dot{N}(t) &= \mu - dN(t) - \beta N(t)(1 - N(t)) + \varepsilon\left(1 - N(t) - C_o(t) - C_r(t)\right), \\
\dot{C}_o(t) &= \beta N(t)(1 - N(t)) - (d + \gamma)C_o(t), \\
\dot{C}_r(t) &= \gamma C_o(t) - (d + \sigma)C_r(t).
\end{aligned} \tag{2}$$

In order to apply the model for describing the dynamics of the cocaine consumption in Spain, we need to estimate the model parameters μ, d, β, ε, γ, σ of (2), that best adjust the information collected in Table 1. To this end, we apply Particle Swarm Optimization (PSO) technique [9]. The values of the estimations of the model parameters are shown in Table 2.

Now, we are going to introduce the delay in the nonlinear term of the model (2). This term represents the transmission of the cocaine misuse habit between individuals, being this transition not instantaneous but with a certain lag. However, to the best of our knowledge the size of this delay is unknown. Thus, we introduce the delay via an integral modeling the possibility of becoming a cocaine consumer by contact with consumers in the last months. Using

Model parameters	Estimations
μ	$1.587198\ 10^{-3}$
d	$1.587198\ 10^{-3}$
β	$5.013946\ 10^{-3}$
ε	$5.855882\ 10^{-6}$
γ	$1.003084\ 10^{-3}$
σ	$1.137033\ 10^{-3}$

Table 2: Values of the parameters that best fit model (2) with the data in Table 1 using PSO algorithm [9]. Recall that we assumed that $\mu = d$.

the approach presented in [14], the system of non-linear differential equations (2) is formulated as

$$\begin{aligned}
\dot{N}(t) &= \mu - dN(t) - \beta N(t)\left(1 - \int_0^\infty N(t-s)dK(s)\right) + \varepsilon(1 - N(t) - C_o(t) - C_r(t)),\\
\dot{C}_o(t) &= \beta N(t)\left(1 - \int_0^\infty N(t-s)dK(s)\right) - (d+\gamma)C_o(t),\\
\dot{C}_r(t) &= \gamma C_o(t) - (d+\sigma)C_r(t),
\end{aligned} \tag{3}$$

where $K(s)$ is a non-decreasing function such that $\int_0^\infty dK(s) = 1$ and the integrals should be understood in the Stieltjes sense.

3 Existence of equilibrium points

In dealing with the mathematical models formulated via non-linear differential equations, a main goal is the analysis of equilibrium of the solution. The interest of this objective is justified because slight changes in the model inputs may lead to important deviations in the model output resulting in inaccurate solutions.

Now, we determine the equilibrium $(N^*, C_o^*, C_r^*, C_b^*)$ of (3), by imposing that $\dot{N}(t) = \dot{C}_o(t) = \dot{C}_r(t) = 0$ in (3). This leads to

$$\begin{aligned}
0 &= \mu + \varepsilon(1 - N^* - C_o^* - C_r^*) - dN^* - \beta N^*(1 - N^*),\\
0 &= \beta N^*(1 - N^*) - (d+\gamma)C_o^*,\\
0 &= \gamma C_o^* - (d+\sigma)C_r^*,\\
C_b^* &= 1 - N^* - C_o^* - C_b^*.
\end{aligned} \tag{4}$$

From the third equation in (4) we can obtain C_r^* in terms of C_o^*.

$$C_r^* = cC_o^*, \qquad c := \frac{\gamma}{d+\sigma}. \tag{5}$$

If we add the two first equations in (4) and we apply (5), one gets

$$\mu + \varepsilon - [\varepsilon(1+c) + d + \gamma]C_o^* = (d+\varepsilon)N^*.$$

Thus, we can obtain the value of N^* in terms of C_o^*

$$N^* = a - bC_o^*, \qquad a = \frac{\mu + \varepsilon}{d + \varepsilon}, \qquad b = 1 + \frac{\gamma + c\varepsilon}{d + \varepsilon}. \tag{6}$$

Substituting the expression of N^*, obtained in (6), into the second equation in (4) one gets

$$0 = \beta b^2 (C_o^*)^2 - (\beta b(2a - 1) - d - \gamma) C_o^* + \beta a(a - 1). \tag{7}$$

In order to obtain the values of C_o^* we need to solve the quadratic algebraic equation, (7). To simplify it, note that its discriminant is given by

$$\begin{aligned} D &= (\beta b(2a - 1) - d - \gamma)^2 - 4\beta^2 b^2 a(a - 1) \\ &= \beta^2 b^2 [(2a - 1)^2 - 4a(a - 1)] - 2\beta b(2a - 1)(d + \gamma) + (d + \gamma)^2 \\ &= \beta^2 b^2 - 2\beta b(2a - 1)(d + \gamma) + (d + \gamma)^2 \\ &= \beta^2 b^2 + 2\beta b(d + \gamma) + (d + \gamma)^2 - 4a\beta b(d + \gamma) \\ &= (\beta b + d + \gamma)^2 - 4a\beta b(d + \gamma). \end{aligned}$$

So, the values of C_o^* are given by

$$C_o^* = \frac{\beta b(2a - 1) - (d + \gamma) \pm \sqrt{(\beta b + d + \gamma)^2 - 4\beta ab(d + \gamma)}}{2\beta b^2}. \tag{8}$$

Summarizing, the equilibrium points are given by

$$\begin{aligned} C_r^* &= cC_o^* \\ N^* &= a - bC_o^* \\ C_o^* &= \frac{\beta b(2a - 1) - (d + \gamma) \pm \sqrt{(\beta b + d + \gamma)^2 - 4\beta ab(d + \gamma)}}{2\beta b^2}, \\ C_b^* &= 1 - N^* - C_o^* - C_r^*, \end{aligned} \tag{9}$$

where

$$a = \frac{\mu + \varepsilon}{d + \varepsilon}, \qquad b = 1 + \frac{\gamma + c\varepsilon}{d + \varepsilon}, \qquad c = \frac{\gamma}{d + \sigma}. \tag{10}$$

4 Stochastic perturbations, centering and linearization

As it has been explained in Section 1, there are complex factors having a considerable influence on cocaine consumption. The nature of such factors are mainly random and until now they have not been considered in the model previously formulated. Consequently, the equilibria of (3) will be affected by this uncertainty. This section is devoted to consider this randomness in the mathematical model formulation. In the introduction Section we have justified that the dynamics of cocaine consumption is subject to numerous independent random factors. The central Limit Theorem justifies Gaussian distribution as a suitable pattern to model this kind of uncertainties. This motivates assuming that the deviation of the steady state (N^*, C_o^*, C_r^*) from the current state $(N(t), C_o(t), C_r(t))$ is proportionally affected by perturbations via a Gaussian stochastic process, namely the so called white noise $(\dot{w}_1(t), \dot{w}_2(t), \dot{w}_3(t))$. Specifically, using the approach proposed in [14], this permits the formulate the following model based upon stochastic differential equations of Itô-type.

We will assume that system (3) is exposed to stochastic perturbations of the white noise type, hence Gaussian, that we will denote by $(\dot{w}_1(t), \dot{w}_2(t), \dot{w}_3(t))$ which are directly proportional to the deviation of the system state at $(N(t), C_o(t), C_r(t))$ from the equilibrium point (N^*, C_o^*, C_r^*), given by (9)-(10). This leads to consider the following system of Itô's stochastic differential equations [5]

$$
\begin{aligned}
\dot{N}(t) =& \mu - dN(t) - \beta N(t)\left(1 - \int_0^\infty N(t-s)dK(s)\right) \\
&+ \varepsilon(1 - N(t) - C_o(t) - C_r(t)) + \sigma_1(N(t) - N^*)\dot{w}_1(t), \\
\dot{C}_o(t) =& \beta N(t)\left(1 - \int_0^\infty N(t-s)dK(s)\right) - (d+\gamma)C_o(t) + \sigma_2(C_o(t) - C_o^*)\dot{w}_2(t), \\
\dot{C}_r(t) =& \gamma C_o(t) - (d+\sigma)C_r(t) + \sigma_3(C_r(t) - C_r^*)\dot{w}_3(t),
\end{aligned}
\tag{11}
$$

where $\sigma_i > 0$, $i = 1, 2, 3$, denote constant levels of noise to be determined later and $w_i(t)$, $i = 1, 2, 3$, are mutually independent standard Wiener processes [14, 5].

Now, we centralize the system (11) around the equilibria obtained in Eqs. (9). Note that the equilibrium points in both systems, (3) and (11), are the same. First of all, we center the unknowns with respect to the equilibrium points using the following transformation.

$$
y_1(t) = N(t) - N^*, \quad y_2(t) = C_o(t) - C_o^*, \quad y_3(t) = C_r(t) - C_r^*.
$$

Substituting this into (11) and using (4) we obtain

$$
\begin{aligned}
\dot{y}_1(t) =& \mu - d(y_1(t) + N^*) - \beta(y_1(t) + N^*)\left(1 - \int_0^\infty (y_1(t-s) + N^*)dK(s)\right) \\
&+ \varepsilon(1 - N^* - C_o^* - C_r^*) - \varepsilon(y_1(t) + y_2(t) + y_3(t)) + \sigma_1 y_1(t)\dot{w}_1(t) \\
=& \mu - dN^* - \beta N^*(1 - N^*) - dy_1(t) - \beta y_1(t) + \beta N^* y_1(t) + \beta(y_1(t) + N^*)\int_0^\infty y_1(t-s)dK(s) \\
&+ \varepsilon(1 - N^* - C_o^* - C_r^*) - \varepsilon(y_1(t) + y_2(t) + y_3(t)) + \sigma_1 y_1(t)\dot{w}_1(t) \\
=& - dy_1(t) - \beta y_1(t) + \beta N^* y_1(t) + \beta N^* \int_0^\infty y_1(t-s)dK(s) + \beta y_1(t)\int_0^\infty y_1(t-s)dK(s) \\
&- \varepsilon(y_1(t) + y_2(t) + y_3(t)) + \sigma_1 y_1(t)\dot{w}_1(t) \\
=& - (d + \beta(1 - N^*) + \varepsilon)y_1(t) - \varepsilon(y_2(t) + y_3(t)) \\
&+ \beta N^* \int_0^\infty y_1(t-s)dK(s) + \beta y_1(t)\int_0^\infty y_1(t-s)dK(s) + \sigma_1 y_1(t)\dot{w}_1(t),
\end{aligned}
$$

$$
\begin{aligned}
\dot{y}_2(t) =&\beta(y_1(t)+N^*)\left(1-\int_0^\infty (y_1(t-s)+N^*)dK(s)\right)-(d+\gamma)(y_2(t)+C_o^*)+\sigma_2 y_2(t)\dot{w}_2(t)\\
=&\beta N^*(1-N^*)-\beta N^*\int_0^\infty y_1(t-s)dK(s)-(d+\gamma)y_2-(d+\gamma)C_o^*\\
&+\beta N^*(1-N^*)y_1-\beta y_1\int_0^\infty y_1(t-s)dK(s)+\sigma_2 y_2(t)\dot{w}_2(t)\\
=&\beta N^*(1-N^*)y_1-(d+\gamma)y_2-\beta N^*\int_0^\infty y_1(t-s)dK(s)-\beta y_1\int_0^\infty y_1(t-s)dK(s)+\sigma_2 y_2(t)\dot{w}_2(t),\\
\dot{y}_3(t) =&\gamma(y_2(t)+C_o^*)-(d+\sigma)(y_3(t)+C_r^*)+\sigma_3 y_3(t)\dot{w}_3(t)\\
=&\gamma y_2(t)-dy_3(t)+\sigma_3 y_3(t)\dot{w}_3(t),
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
\dot{y}_1(t) =&-(d+\beta(1-N^*)+\varepsilon)y_1(t)-\varepsilon(y_2(t)+y_3(t))\\
&+\beta N^*\int_0^\infty y_1(t-s)dK(s)+\beta y_1(t)\int_0^\infty y_1(t-s)dK(s)\\
&+\sigma_1 y_1(t)\dot{w}_1(t),\\
\dot{y}_2(t) =&\beta N^*(1-N^*)y_1-(d+\gamma)y_2-\beta N^*\int_0^\infty y_1(t-s)dK(s)\\
&-\beta y_1\int_0^\infty y_1(t-s)dK(s)+\sigma_2 y_2(t)\dot{w}_2(t),\\
\dot{y}_3(t) =&\gamma y_2(t)-dy_3(t)+\sigma_3 y_3(t)\dot{w}_3(t).
\end{aligned}
\tag{12}
$$

It is clear that stability of the equilibrium of the system (11) is equivalent to stability of the zero solution of system (12).

Rejecting the non-linear terms in (12), we obtain the linear part of the system (12)

$$
\begin{aligned}
\dot{z}_1(t) =&-(d+\beta(1-N^*)+\varepsilon)z_1(t)-\varepsilon(z_2(t)+z_3(t))\\
&+\beta N^*\int_0^\infty z_1(t-s)dK(s)+\sigma_1 z_1(t)\dot{w}_1(t),\\
\dot{z}_2(t) =&\beta N^*(1-N^*)z_1-(d+\gamma)z_2-\beta N^*\int_0^\infty z_1(t-s)dK(s)+\sigma_2 z_2(t)\dot{w}_2(t),\\
\dot{z}_3(t) =&\gamma z_2(t)-dz_3(t)+\sigma_3 z_3(t)\dot{w}_3(t).
\end{aligned}
\tag{13}
$$

5 Studying the steady state

Now we shall provide conditions so that the null solution of (12) is stable in probability. To this end, via [14, Remark 5.3], it is enough to establish asymptotic mean square stability of the zero solution of the linear system (13) that is the linear part of the non-linear system (12).

Putting $z(t) = (z_1(t), z_2(t), z_3(t))^T$, the system (13) can be rewritten in the following matrix form

$$\dot{z}(t) = Az(t) + \int_0^\infty Bz(t-s)dK(s) + \sum_{i=1}^{3} C_i z(t)\dot{w}_i(t), \tag{14}$$

where the matrix $C_i = [c_{i,i} = \sigma_i]$ $i = 1, 2, 3$ has the element $c_{ii} = \sigma_i$, $i = 1, 2, 3$, and all other elements are zeros, and

$$A = \begin{pmatrix} -(d + \beta(1 - N^*) + \varepsilon) & -\varepsilon & -\varepsilon \\ \beta N^*(1 - N^*) & -(d + \gamma) & 0 \\ 0 & \gamma & -d \end{pmatrix}, \quad B = \begin{pmatrix} \beta N^* & 0 & 0 \\ -\beta N^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The following theorem gives a sufficient condition for asymptotic mean square stability of the zero solution of the linear stochastic differential equation (14). This condition is derived using the Liapunov's method introduced in [14] by taking the advantage of Linear Matrix Inequalitites (LMIs) [4].

Theorem 5.1 *Let us assume that for some positive definite matrices $P, R \in R^{3\times 3}$ the following LMI*

$$\Psi_0 = \begin{bmatrix} PA + A'P + \sum_{j=1}^{3} C_j' P C_j + R & PB \\ * & -R \end{bmatrix} < 0 \tag{15}$$

holds. Then, the zero solution of the equation (14) is asymptotically mean square stable and the equilibrium $E = (N^*, C_o^*, C_r^*, C_b^*)$ of the system (11) is stable in probability.

Example 5.1 *Using the values of the parameters from the Table 2, we have $c = 0.3682$, $a = 1$ and $b = 1.6310$, and as a consequence, $\beta b > d + \gamma$. Therefore, since $a = 1$, from (8) it follows*

$$C_o^* = \frac{\beta b - (d+\gamma) \pm \sqrt{(\beta b - (d+\gamma))^2}}{2\beta b^2},$$

i.e,

$$C_{o1}^* = 0, \qquad C_{o2}^* = \frac{\beta b - (d+\gamma)}{\beta b^2}.$$

So, we obtain two equilibria: $E_0 = (N^*, C_o^*, C_r^*, C_b^*) = (1, 0, 0, 0)$ and $E_1 = (N^*, C_o^*, C_r^*, C_b^*) = (0.3167, 0.4189, 0.1542, 0.1101)$. Using MATLAB and Theorem 5.1, it is shown that the equilibrium E_0 is unstable and the equilibrium E_1 is stable in probability by the following levels of noises: $\sigma_1 = 0.08$, $\sigma_2 = 0.07$, $\sigma_3 = 0.05$.

6 Building simulations from real data

Once the theoretical analysis of the stochastic model has been addressed, we shall apply the model to check whether is suitable to describe the dynamics of cocaine consumption in Spain. To this end, we will construct numerical simulations of model (11) using the figures reported in Table 2 and Example 5.1 for the model parameters $(\mu, d_1, \beta, \epsilon, \gamma, \sigma_1, \sigma_2, \sigma_3)$ and the endemic steady state $E = (N^*, C_o^*, C_r^*, C_b^*)$. To perform these simulations, we fix the delay, $h > 0$, and we take into account that $dK(s) = \delta(s-h)\mathrm{d}s$, being $\delta(s)$ the Dirac delta function. The continuous model (11) has been discretized using the Euler-Maruyama scheme [14, pp. 309-310]. This yields:

$$\begin{aligned} N_{i+1} &= N_i + \Delta t \left(\mu - dN_i - \beta N_i(1-N_{i-m}) + \varepsilon(1 - N_i - C_{o,i} - C_{r,i})\right) \\ &\quad + \sigma_1 \left(N_i - N^*\right)\left(W_{1,i+1} - W_{1,i}\right) \\ C_{o,i+1} &= C_{o,i} + \Delta t \left(\beta N_i(1-N_{i-m}) - (d+\gamma)C_{o,i}\right) \\ &\quad + \sigma_2 \left(C_{o,i} - C_o^*\right)\left(W_{2,i+1} - W_{2,i}\right) \\ C_{r,i+1} &= C_{r,i} + \Delta t \left(\gamma C_{o,i} - (d+\sigma)C_{r,i}\right) \\ &\quad + \sigma_3 \left(C_{r,i} - C_r^*\right)\left(W_{3,i+1} - W_{3,i}\right), \end{aligned} \tag{16}$$

where Δt is the discretization time step, $m = h/\Delta t$ is the discretized delay, $N_i = N(t_i)$, $C_{o,i} = C_o(t_i)$, $C_{r,i} = C_r(t_i)$, $t_i = i\Delta t$, $i = 0, 1, 2, \ldots$ and $W_{k,i} = W_k(t_i)$, $k = 1, 2, 3$, are the values of the Brownian motion of standard Wiener process at t_i. These values are obtained via simulations (see [14, Section 2.1.1]).

In particular, we have considered the discretization time step as one month ($\Delta t = 1$) and as the delay to become a cocaine consumer until one year, i.e, 12 months ($h = 12$), then $m = h = 12$. This value of m is assumed, and it means that in any moment during a year, a non-cocaine consumer may become a cocaine consumer due to peer pressure.

As $m = 12$, in order to run the numerical scheme, we need setting values of the previous 12 months, from Dec 2000 to Nov 2001. As these values are not available, we use a numerical backward approach [15] in the discretized stochastic system (16) in order to set $N_i, C_{o,i}, C_{r,i}$ from $N_{i+1}, C_{o,i+1}, C_{r,i+1}$. This backward process starts with the data corresponding to Dec 2001 (see first column of Table 1) and ends in Dec 2000.

In Figure 2, we have represented 500 simulations of the discretized model given in (16). From this graphical representation, we can see that the numerical simulations converge, in the long-run, to the endemic steady state calculated in Example 5.1, i.e. $E_1 = (N^*, C_o^*, C_r^*, C_b^*) = (0.3167, 0.4189, 0.1542, 0.1101)$ (dashed line).

7 Conclusions

In this paper we have proposed a stochastic model based on a non-linear system of differential equations with delay to describe the dynamics of cocaine con-

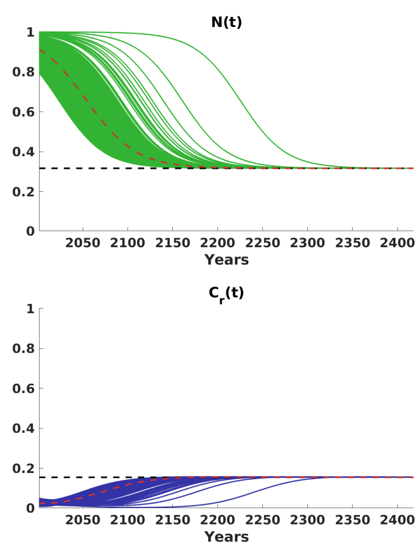

Figure 2: Simulations of 500 trajectories of the approximated solution stochastic process modelling the dynamics of cocaine consumption in Spain according to stochastic system with delay (11). Those approximations have been constructed using the numerical scheme (16) taking $\Delta t = 1$ month and delay $h = 12$ months. Red line represents the average of the trajectories and the black one represents the equilibrium point, $E_1 = (N^*, C_o^*, C_r^*, C_b^*) = (0.3167, 0.4189, 0.1542, 0.1101)$.

sumption. We have applied appropriate tools in order to establish conditions so that the stochastic stability of the steady state is guaranteed. Furthermore, we have carried out simulations of the proposed model that are in full agreement with real data. These simulations have been used to predict the long-run behaviour of cocaine consumption in Spain. The stochastic model approach may be useful to Spanish Health Ministry policy makers to designing future strategies based upon the knowledge about the consumption of cocaine in Spain on the long term provided the current health policy does not change. Moreover, taking into account the interpretation of the model parameters and their relationship with specific health campaigns detailed in [11], the model can be also applied to quantify the effect of health campaigns to reduce the consumption of cocaine on the long term. For instance, and with the aim of being more illustrative, if policy makers decide to apply a specific health campaign addressed to the subpopulation, say C_b (habitual cocaine consumers), and afterwards the data of cocaine consumption is collected (similarly to Table 1), then the model parameters can be determined using our approach and, the model will permit to predict how the long term behaviour of subpopulation C_b will change. In this manner, the model may be useful to have a picture of the effect of the implemented health campaign.

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