On σ-subnormal subgroups of factorised finite groups

A. Ballester-Bolinches, S.F. Kamornikov, M.C. Pedraza-Aguilera and X. Yi

Abstract

Let $\sigma = \{\sigma_i : i \in I\}$ be a partition of the set $\mathbb{P}$ of all prime numbers. A subgroup X of a finite group G is called *σ-subnormal* in G if there is chain of subgroups

$$X = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = G$$

with X_{i-1} normal in X_i or $X_i/Core_{X_i}(X_{i-1})$ is a σ_i-group for some $i \in I$, $1 \leq i \leq n$.

In the special case that σ is the partition of $\mathbb{P}$ into sets containing exactly one prime each, the σ-subnormality reduces to the familiar case of subnormality.

If a finite soluble group $G = AB$ is factorised as the product of the subgroups A and B, and X is a subgroup of G such that X is σ-subnormal in $\langle X, X^g \rangle$ for all $g \in A \cup B$, we prove that X is σ-subnormal in G. This is an extension of a subnormality criteria due to Maier and Sidki and Casolo.
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1 Introduction and statements of results.

All groups considered in this paper are finite.

An important subnormality criterion asserts that if $G = AB$ is a group which is the product of the subgroups A and B and X is a subgroup of G contained in $A \cap B$ that is subnormal in A and B, then X is subnormal in G. This result was proved by Maier in [5] for soluble groups and then for arbitrary groups by Wielandt [11]. In the same paper, Wielandt conjectured that if X is a subgroup of G such that X is subnormal in $\langle X, X^g \rangle$ for all $g \in A \cup B$, then X is subnormal in G.

Wielandt's conjecture was proved to be true in the soluble universe by Maier and Sidki [6] for subgroups of prime power order and then for arbitrary soluble groups by Casolo in [2].

Theorem 1. *Let the soluble group $G = AB$ be the product of the subgroups A and B. If X is a subgroup of G such that X is subnormal in $\langle X, X^g \rangle$ for all $g \in A \cup B$, then X is subnormal in G.*

Recently, Skiba [7] has generalised the concept of subnormality, introducing σ-subnormality, in which σ is a partition of the set $\mathbb{P}$, the set of all primes. Hence $\mathbb{P} = \bigcup_{i \in I} \sigma_i$, with $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$.

A group G is *σ-primary* if the prime factors, if any, of its order all belong to the same member of σ.

Definition 1. *A subgroup X of a group G is called σ-subnormal in G if there is chain of subgroups*

$$X = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = G$$

with X_{i-1} normal in X_i or $X_i/Core_{X_i}(X_{i-1})$ σ-primary for $1 \leq i \leq n$.

In the special case that σ is the partition of $\mathbb{P}$ into sets containing exactly one prime each, the σ-subnormality reduces to the familiar case of subnormality.

Skiba [7] has also extended the concept of nilpotency, introducing σ-nilpotency.

Definition 2. *A group G is said to be* σ-nilpotent *if it is a direct product of σ-primary groups.*

He proved that the class $\mathcal{N}_\sigma$ of all σ-nilpotent groups is a subgroup-closed saturated Fitting formation ([7, Corollary 2.4 and Lemma 2.5]).

Note that a subgroup X of a group G is σ-subnormal in G if and only if it is K-$\mathcal{N}_\sigma$-subnormal in G in the sense of [1, Definition 6.1]. This characterisation allows us to deduce that *the set of all σ-subnormal subgroups of a group G forms a sublattice of the subgroup lattice of G* ([1, Lemmas 6.3.11 and 6.3.12

and Example 6.3.13]). This will be used in the paper without any further reference.

Skiba [7] showed that the set of all σ-subnormal subgroups has a strong influence on the structure of σ-soluble groups, that is, groups in which every chief factor is σ-primary. Therefore it is natural and interesting to investigate which of remarkable theorems about subnormal subgroups have analogs when we consider σ-subnormal subgroups of σ-soluble groups.

Our objective in this paper is to present a σ-subnormal version of Theorem 1. We prove:

Theorem A. *Assume that G is a soluble group factorised as a product $G = AB$ of the subgroups A and B. Let X be a subgroup of G such that X is σ-subnormal in $\langle X, X^g \rangle$ for all $g \in A \cup B$. Then X is σ-subnormal in G.*

Theorem 1 is just Theorem A for the partition of $\mathbb{P}$ into sets containing exactly one prime each.

We shall adhere to the notation and terminology of [1] and [3].

2 Preparatory lemmas

In this section we collect some results which will be used in the proof of our main theorem.

Our first lemma collects some basic properties of σ-subnormal subgroups.

Lemma 1 ([7]). *Let H, K and N be subgroups of a group G. Suppose that H is σ-subnormal in G and N is normal in G. Then the following statements hold:*

1. *$H \cap K$ is σ-subnormal in K.*
2. *If K is a σ-subnormal subgroup of H, then K is σ-subnormal in G.*
3. *If K is σ-subnormal in G, then $H \cap K$ is σ-subnormal in G.*
4. *HN/N is σ-subnormal in G/N.*
5. *If $N \subseteq K$ and K/N is σ-subnormal in G/N, then K is σ-subnormal in G.*

6. *If $K \subseteq H$ and H is σ-nilpotent, then K is σ-subnormal in G.*

7. *If H is a σ_i-group, where $\sigma_i \in \sigma$, then $H \leq \mathrm{O}_{\sigma_i}(G)$.*

According to a result of Wielandt (see [4, Theorem 7.3.3]), given a subgroup X of a group G, it is enough to know that X is subnormal in the subgroup generated by $\langle X, X^g \rangle$ for all $g \in G$ to deduce that X is subnormal in G. This result still holds for σ-subnormal subgroups of soluble groups and it is a direct consequence of [1, Proposition 6.1.10 and Theorem 6.2.17]. We include a proof here for the sake of completeness.

Lemma 2. *Suppose that G is a soluble group and X is a subgroup of G such that it is σ-subnormal in $\langle X, X^x \rangle$ for all $x \in G$. Then X is σ-subnormal in G.*

Proof. We argue by induction on $\mid G \mid$. If G is σ-nilpotent, then X is σ-subnormal in G by Lemma 1 (6). Hence we may assume that G is not σ-nilpotent so that the σ-nilpotent residual L of G is non-trivial. Let N be a minimal normal subgroup of G such that $N \leq L$. By inductive hypothesis, XN/N is σ-subnormal in G/N and so XN is σ-subnormal in G by Lemma 1 (5). If XN were a proper subgroup of G, then X would be σ-subnormal in XN by the inductive hypothesis. Applying Lemma 1 (2), it follows that X is σ-subnormal in G. Hence we can suppose that $G = XN$ and $G \neq X$. Since N is abelian, X is a maximal subgroup of G. If X is normal in G, then it is σ-subnormal in G. If X is not normal in G, there exists an element $g \in G$ such that $X \neq X^g$. Then $G = \langle X, X^g \rangle$ and X is σ-subnormal in G. The proof is now complete. □

Our next lemma is crucial in the proof of our main result. It confirms that Theorem A holds for σ-primary subgroups.

Lemma 3. *Let the soluble group $G = AB$ be the product of the subgroups A and B. Let X be a σ-primary subgroup of G such that X is σ-subnormal in $\langle X, X^g \rangle$ for all $g \in A \cup B$, then X is σ-subnormal in G.*

Proof. Suppose that the result is false. We choose a counterexample G of minimal order and proceed to derive a contradiction. Without loss of generality, we may assume that X is a σ_1-group. We deduce

1. $\mathrm{O}_{\sigma_1}(G) = 1$.

Suppose that $T = \mathrm{O}_{\sigma_1}(G) \neq 1$. Observe that XT/T satisfies the hypothesis of the lemma in $G/T = (AT/T)(BT/T)$ by Lemma 1 (4). By the minimality of G we have that XT/T is σ-subnormal in G/T. Applying Lemma 1 (5), we have that XT is σ-subnormal in G. Since XT is a σ_1-group, X is σ-subnormal in G by Lemma 1 (2) and (6), which is not the case.

2. There exists a prime $p \notin \sigma_1$ such that G is a $(\sigma_1 \cup \{p\})$-group.

   Since G is soluble and $\mathrm{O}_{\sigma_1}(G) = 1$, there exists a prime $p \notin \sigma_1$ such that $\mathrm{O}_p(G) \neq 1$. Let N be a minimal normal subgroup of G contained in $\mathrm{O}_p(G)$. Then XN/N satisfies the hypothesis of the lemma in $G/N = (AN/N)(BN/N)$ by Lemma 1 (4). By minimality of G, XN/N is σ-subnormal in G/N. Since XN/N is a σ_1-group, we can apply Lemma 1 (7) to conclude that $XN/N \leq O_{\sigma_1}(G/N) = O/N$. Clearly O is a normal $(\sigma_1 \cup \{p\})$-subgroup of G containing X. Thus $X \leq O \leq \mathrm{O}_{\sigma_1 \cup \{p\}}(G)$. Let A_1 and B_1 be Hall $(\sigma_1 \cup \{p\})$-subgroups of A and B respectively such that $H = A_1B_1$ is a Hall $(\sigma_1 \cup \{p\})$-subgroup of G. Since $\mathrm{O}_{\sigma_1 \cup \{p\}}(G)$ is normal in G, we have that $\mathrm{O}_{\sigma_1 \cup \{p\}}(G) \leq H$. If H is a proper subgroup of G, then the minimality of G implies that X is σ-subnormal in H. By Lemma 1 (1), X is σ-subnormal in $\mathrm{O}_{\sigma_1 \cup \{p\}}(G)$. Since $\mathrm{O}_{\sigma_1 \cup \{p\}}(G)$ is σ-subnormal in G, it follows that X is σ-subnormal in G by Lemma 1 (2), which is not so by hypothesis. Hence $G = H$ is a $(\sigma_1 \cup \{p\})$-group.

3. G has a unique minimal normal subgroup.

   Suppose that G has two different minimal normal subgroups, N_1 and N_2 say. Arguing as in Step 2, XN_2/N_2 is σ-subnormal σ_1-group of G and $XN_2/N_2 \leq \mathrm{O}_{\sigma_1}(G/N_2) = K/N_2$. Assume that $[X, N_1] \neq 1$. Since $[X, N_1] \leq [XN_2, N_1] \leq K \cap N_1$, it follows that $N_1 \leq K$. Therefore N_1 is a σ_1-group, which is not so by Step (1). Thus we may assume that $[X, N_1] = [X, N_2] = 1$. Then X is a normal subgroup of XN_2 which is a σ-subnormal subgroup of G by Lemma 1 (5). Applying Lemma 1 (2), it follows that X is σ-subnormal in G, which is not the case.

4. G is a primitive group. Therefore $N = \mathrm{Soc}(G) = \mathrm{C}_G(N) = \mathrm{F}(G) = \mathrm{O}_p(G)$ is the unique minimal normal subgroup of G which is complemented by a core-free maximal subgroup of G.

Let $N = \mathrm{Soc}(G)$ be the unique minimal normal subgroup of G. Write $\mathrm{O}_{\sigma_1}(G/N) = K/N$. Suppose that $N \leq \Phi(G)$, it follows by [3, A, Lemma 13.2], that $K = N \times K_{\sigma_1}$, where K_{σ_1} denotes the Hall σ_1-subgroup of K. Arguing as in Step 3, we have that $XN \leq K$. Hence $X \leq K_{\sigma_1}$. But K_{σ_1} is a subnormal σ_1-subgroup of G. By Lemma 1 (7), $K_{\sigma_1} \leq \mathrm{O}_{\sigma_1}(G) = 1$. This is a contradiction. Therefore $\Phi(G) = 1$, G is primitive and N is complemented by a core-free maximal subgroup of G. By [1, Theorem 1.1.7], $N = \mathrm{C}_G(N) = \mathrm{F}(G)$. By Steps 1 and 2, $N = \mathrm{O}_p(G)$.

5. $N \cap A = N \cap B = 1$.

   Assume that $N \cap A \neq 1$ and let $1 \neq t \in N \cap A$. Then $\langle X, X^t \rangle = [t, X]X$. Since X is σ-subnormal in $\langle X, X^t \rangle$, we have that X^t is σ-subnormal in $\langle X, X^t \rangle^t$. But $\langle X, X^t \rangle^t = ([t, X]X)^t = [t, X]^t X^t = [t, X]X^t \leq [t, X][t, X]X = [t, X]X$. Therefore $\langle X, X^t \rangle^t = \langle X, X^t \rangle$ and X^t is also a σ-subnormal subgroup of $\langle X, X^t \rangle$. Then, X and X^t are σ-subnormal σ_1-subgroups of $\langle X, X^t \rangle$. By Lemma 1 (7), $\langle X, X^t \rangle \leq \mathrm{O}_{\sigma_1}(\langle X, X^t \rangle)$. Thus $\langle X, X^t \rangle$ is a σ_1-group. Furthermore, $[t, X] \leq [N, X] \leq N$ is a p-group. Therefore $[t, X] = 1$ and $X \leq \mathrm{C}_G(N \cap A) \leq \mathrm{N}_G(N \cap A) = H$. Hence $N \cap A \neq N$ by Step 4 and so $H = A(H \cap B)$ is a proper subgroup of G. The minimal choice of G implies that X is σ-subnormal in H. By Lemma 1 (7), $X \leq \mathrm{O}_{\sigma_1}(H)$. Since $N \leq H$ and N is a ${\sigma_1}'$-group, we have that X centralizes N, which is not so by Step 4.

6. Let P be a Sylow p-subgroup of G such that $P = (P \cap A)(P \cap B)$. Then $|N| \leq |P \cap A|$.

   Since $N(P \cap B) \leq P$, and $N \cap B = 1$, we have

   $$|N||(P \cap B)|/|N \cap (P \cap B)| = |N||(P \cap B)|$$

   and

   $$|N||(P \cap B)| \leq |(P \cap A)||(P \cap B)|/|P \cap A \cap B| \leq |P \cap A||P \cap B|.$$

   Thus $|N| \leq |P \cap A|$.

7. We have a contradiction.

   By Step 6, $|N| \leq |P \cap A|$ and, by Step 5, $|N||(P \cap A)| \leq |P|$. Therefore $|N|^2 \leq |P|$.

Let M be a core-free maximal subgroup of G such that $G = NM$ and $N \cap M = 1$. Then $P = NM_p$, where $M_p = P \cap M$ is a Sylow p-subgroup of M. It is clear that N is a faithful and irreducible M-module over the finite field of p-elements. Applying [12, Corollary 1.9], we have that $|M_p| \leq |N|/2$. Thus $|P| \leq |N|^2/2$. Consequently, $2|N|^2 \leq 2|P| \leq |N|^2$, which is not the case.

□

3 Proof of Theorem A

Proof. Suppose that the theorem is false and let G be a counterexample for which $|G| + |X|$ is minimal. Since the join of σ-subnormal subgroups is again σ-subnormal, it follows by Lemma 3 that X is not σ-nilpotent. Let $1 \neq X^{\mathcal{N}_\sigma}$ be the $\mathcal{N}_\sigma$-residual of X and let $S/X^{\mathcal{N}_\sigma}$ be a maximal subgroup of $X/X^{\mathcal{N}_\sigma}$. Then, by Lemma 1 (5) and (6), S is a maximal subgroup of X and S is σ-subnormal in X. By Lemma 1 (1) and (2), S is σ-subnormal in $\langle S, S^g \rangle$ for all $g \in A \cup B$. By the minimal choice of the pair (G, X), we have that S is σ-subnormal in G. Suppose that $T/X^{\mathcal{N}_\sigma}$ is another maximal subgroup of $X/X^{\mathcal{N}_\sigma}$. Then T is is σ-subnormal in G and therefore $X = \langle S, T \rangle$ is σ-subnormal in G, contrary to supposition. Therefore $X/X^{\mathcal{N}_\sigma}$ has a unique maximal subgroup and so $X/X^{\mathcal{N}_\sigma}$ is a cyclic p-group for some prime p. Without loss of generality we may assume that $p \in \sigma_1$.

By [1, Lemma 6.1.9(1)], $X^{\mathcal{N}_\sigma}$ is subnormal in $\langle X, X^g \rangle$ for all $g \in A \cup B$. Hence $X^{\mathcal{N}_\sigma}$ is subnormal in $\langle X^{\mathcal{N}_\sigma}, (X^{\mathcal{N}_\sigma})^g \rangle$ for all $g \in A \cup B$. Applying Theorem 1, we conclude that $X^{\mathcal{N}_\sigma}$ is subnormal in G.

Assume that $X \leq A$. Then, by Lemma 2, X is σ-subnormal in A. In particular, $A \neq G$. Let M be a maximal subgroup of G containing A. Then $M = M \cap AB = A(M \cap B)$ and M satisfies the hypotheses of the theorem. By the choice of G, X is σ-subnormal in M. Write $K = \mathrm{Core}_G(M)$. Observe that XK/K satisfies the hypothesis of the theorem in $G/K = (AK/K)(BK/K)$ by Lemma 1 (4). If $K \neq 1$, the minimality of G implies that XK/K is σ-subnormal in G/K. By Lemma 1 (5), XK is σ-subnormal in G. Since $X \leq XK \leq M$, it follows that X is σ-subnormal in XK by Lemma 1 (1). Hence X is σ-subnormal in G by Lemma 1 (2), contrary to assumption. Therefore $K = 1$ and so G is a primitive group. Then $N = \mathrm{Soc}(G) = \mathrm{F}(G)$ is the unique minimal normal subgroup of G and there exists a maximal subgroup M of G

such that $G = NM$ and $N \cap M = 1$. In particular $N \cap X^{\mathcal{N}_\sigma} \le N \cap M = 1$. Since $X^{\mathcal{N}_\sigma}$ is subnormal in G, we have that $\mathrm{F}(X^{\mathcal{N}_\sigma}) \le \mathrm{F}(G) = N$. Therefore $\mathrm{F}(X^{\mathcal{N}_\sigma}) = 1$. Since $X^{\mathcal{N}_\sigma}$ is soluble, $X^{\mathcal{N}_\sigma} = 1$. Thus X is a σ-nilpotent group, a contradiction.

Suppose that X is not contained in A. Let $A_0 = \langle A, X\rangle$. Then $A_0 = A_0 \cap AB = A(A_0 \cap B)$ and A_0 satisfies the hypotheses of the theorem. If $A_0 \ne G$, then X is σ-subnormal in A_0 by the choice of G. Let $g \in A_0 \cup B$. If $g \in B$, then X is σ-subnormal in $\langle X, X^g\rangle$. If $g \in A_0$, then $X \le \langle X, X^g\rangle \le \langle A, X\rangle = A_0$. Since X is σ-subnormal in A_0, it follows that X is σ-subnormal in $\langle X, X^g\rangle$ by Lemma 1 (1). Hence X is σ-subnormal in $\langle X, X^g\rangle$ for all $g \in A_0 \cup B$. Then the above reasoning implies that X is σ-subnormal in G, which is a contradiction. Thus we may assume that $G = \langle A, X\rangle$. Let q be a prime dividing the order of $\mathrm{F}(X^{\mathcal{N}_\sigma})$ and denote $R = \mathrm{O}_q(G)$. Since $\mathrm{F}(X^{\mathcal{N}_\sigma})$ is subnormal in G, we have that $1 \ne \mathrm{O}_q(X^{\mathcal{N}_\sigma}) \le R$.

Suppose that $q \in \sigma_1$. Let N be a minimal normal subgroup of G contained in R. Then XN/N satisfies the hypothesis of the theorem in $G/N = (AN/N)(BN/N)$ by Lemma 1 (4). By minimality of G, XN/N is σ-subnormal in G/N. Then XN is σ-subnormal in G by Lemma 1 (5). Since $X^{\mathcal{N}_\sigma}$ is subnormal in G, we have that $N \le \mathrm{N}_G(X^{\mathcal{N}_\sigma})$ by [3, Lemma A.14.3]. Then $X^{\mathcal{N}_\sigma}$ is normal in XN and $XN/X^{\mathcal{N}_\sigma} = (X/X^{\mathcal{N}_\sigma})(NX^{\mathcal{N}_\sigma}/X^{\mathcal{N}_\sigma})$ is a σ_1-group. Applying Lemma 1 (6), we conclude that $X/X^{\mathcal{N}_\sigma}$ is a σ-subnormal subgroup of $XN/X^{\mathcal{N}_\sigma}$. By Lemma 1 (2), X is a σ-subnormal subgroup of G, a contradiction.

Then $q \notin \sigma_1$. We may assume that $q \in \sigma_2$. Denote $W = R \cap X^{\mathcal{N}_\sigma}$. Since $1 \ne \mathrm{O}_q(X^{\mathcal{N}_\sigma}) \le R$, it follows that $W \ne 1$. Moreover W is a subnormal subgroup of G by [3, Corollary A.14.2]. Let $g \in A$. Then X is σ-subnormal in $\langle X, X^g\rangle$. In particular X is σ-subnormal in $\langle X, W^g\rangle$ by Lemma 1 (1). Therefore there exists a chain of subgroups

$$X = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = \langle X, W^g\rangle$$

such that either X_{i-1} is normal in X_i or $X_i/\operatorname{Core}_{X_i}(X_{i-1})$ is σ-primary for all $i = 1, 2, \dots, n$. On the other hand, $X \le \langle X, W^g\rangle \le XR^g = XR$. Hence $|\langle X, W^g\rangle : X|$ is a q-number. Then either X_{i-1} is normal in X_i and X_i/X_{i-1} is a q-number or $X_i/\operatorname{Core}_{X_i}(X_{i-1})$ is a σ_2-group for all $1 \le i \le n$.

We prove that $\mathrm{O}^{\sigma_2}(\langle X, W^g\rangle) = \mathrm{O}^{\sigma_2}(X)$. First we see that $\mathrm{O}^{\sigma_2}(\langle X, W^g\rangle) = \mathrm{O}^{\sigma_2}(X_{n-1})$. Clearly $\mathrm{O}^{\sigma_2}(X_{n-1}) \le O^{\sigma_2}(\langle X, W^g\rangle) = \mathrm{O}^{\sigma_2}(X_n)$. Assume that X_{n-1} is a normal subgroup of X_n. Then $\mathrm{O}^{\sigma_2}(X_{n-1})$ is a normal subgroup of

X_n. Therefore $X_n/\operatorname{O}^{\sigma_2}(X_{n-1})$ is a σ_2-group and we have that $\operatorname{O}^{\sigma_2}(X_{n-1}) = \operatorname{O}^{\sigma_2}(X_n)$. Assume that $X_n/\operatorname{Core}_{X_n}(X_{n-1})$ is σ_2-group. Then it follows that $(X_n/\operatorname{O}^{\sigma_2}(\operatorname{Core}_{X_n}(X_{n-1})/\operatorname{Core}_{X_n}(X_{n-1})/\operatorname{O}^{\sigma_2}(\operatorname{Core}_{X_n}(X_{n-1}))$ is isomorphic to $X_n/\operatorname{Core}_{X_n}(X_{n-1})$. Hence $X_n/\operatorname{O}^{\sigma_2}(\operatorname{Core}_{X_n}(X_{n-1}))$ is a σ_2-group. This implies that $\operatorname{O}^{\sigma_2}(X_n) \leq \operatorname{O}^{\sigma_2}(\operatorname{Core}_{X_n}(X_{n-1})) \leq \operatorname{O}^{\sigma_2}(X_{n-1})$. Consequently, $\operatorname{O}^{\sigma_2}(\langle X, W^g\rangle) = \operatorname{O}^{\sigma_2}(X_n) = \operatorname{O}^{\sigma_2}(X_{n-1})$. Arguing by induction we have that $\operatorname{O}^{\sigma_2}(\langle X, W^g\rangle) = \operatorname{O}^{\sigma_2}(X_0) = \operatorname{O}^{\sigma_2}(X) \leq X$.

Since $X/O^{\sigma_2}(\langle X, W^g\rangle)$ is σ-nilpotent, $X^{\mathcal{N}_\sigma} \leq O^{\sigma_2}(\langle X, W^g\rangle)$. Since $X/X^{\mathcal{N}_\sigma}$ is a p-group, and $p \notin \sigma_2$, we have that $\operatorname{O}^{\sigma_2}(\langle X, W^g\rangle) = X$ and then X is a normal subgroup of $\langle X, W^g\rangle$. Since it holds for every element of A, we have that X is a normal subgroup of XW^A. Hence $[X, W^A] \leq X$. Moreover $[X, W^A] \leq [X, R] \leq R$. Therefore $[X, W^A] \leq X \cap R$. Since $(X \cap R)X^{\mathcal{N}_\sigma}/X^{\mathcal{N}_\sigma} = 1$ (notice that it is a p-group, with $p \in \sigma_1$, and a q-group, with $q \in \sigma_2$), we have that $X \cap R = X^{\mathcal{N}_\sigma} \cap R = W$. Thus $[X, W^A] \leq W \leq W^A$ and X normalises W^A. This means that W^A is a normal subgroup of $G = \langle X, A\rangle$. On the other hand, $W^A \neq 1$ since $W \neq 1$. Since XW^A/W^A satisfies the hypothesis of the lemma in $G/W^A = (AW^A/W^A)(BW^A/W^A)$ by Lemma 1 (4), it follows that XW^A/W^A is σ-subnormal in G/W^A by the choice of G. Then XW^A is σ-subnormal in G by Lemma 1 (5). Then the normality of X in XW^A implies that X is σ-subnormal in G by Lemma 1 (2). This contradiction completes the proof.

□

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A. Ballester-Bolinches
Departament de Matemàtiques, Universitat de València
Dr. Moliner 50, 46100 Burjassot, València, Spain

e-mail: Adolfo.Ballester@uv.es

S.F. Kamornikov
Department of Mathematics
F. Scorina Gomel State University
Gomel 246019, Belarus
e-mail: sfkamornikov@mail.ru

M.C. Pedraza-Aguilera
Instituto Universitario de Matemática Pura y Aplicada
Universitat Politècnica de València,
Camino de Vera, 46022, Valencia, Spain
e-mail: mpedraza@mat.upv.es

X. Yi
Department of Mathematics
Zhejiang Sci-Tech University
Hangzhou 310018 Zheji, People's Republic of China
e-mail: yixiaolan2005@126.com