Applied General Topology

Appl. Gen. Topol. 22, no. 1 (2021), 149-167
doi:10.4995/agt.2021.14422

On sheaves of Abelian groups and universality

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Communicated by D. Georgiou

Abstract

Universal elements are one of the most essential parts in research fields, investigating if there exist (or not) universal elements in different classes of objects. For example, classes of spaces and frames have been studied under the prism of this universality property. In this paper, studying classes of sheaves of Abelian groups, we construct proper universal elements for these classes, giving a positive answer to the existence of such elements in these classes.

2010 MSC: *14F05; 18F20; 54B40.*

Keywords: *sheaves; universal sheaves; universal spaces; containing spaces; saturated classes of spaces.*

1. Introduction and Preliminaries

The notion of "universal object" is considered in many branches of Mathematics. The problem of the existence of such objects is naturally arised whenever a new category of objects is appeared. Especially for the branch of Topology, the problem of the existence of universal elements in different classes of topological spaces was considered at the first steps of its development. Now, in the bibliography there are lots of papers concerning universal objects. Many of them are indicated in the book [9].

In the paper [7] and in the above mentioned book, a method of construction of so-called Containing Spaces is developed. This method can be used for the

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Received 29 September 2020 – Accepted 20 December 2020

construction of universal objects in different categories. Such categories are, for example, topological spaces (with different dimension invariants)(see [3], Chapter 3 of [9]), separable metric spaces (see Chapter 9 of [9], [10], [12], [13], [15]), mappings (see [8], Chapter 6 of [9], [10], [11]), topological groups ([11], [14], [17]), G-spaces (see Chapter 7 of [9], [10], [11], [17], [18]) and frames (see [2], [4], [5], [6], [16]).

In the present paper we use this method for construction of universal objects in the category of sheaves of Abelian groups, which play an important role in the study of cohomology theories of general topological spaces.

General notation and assumptions. An ordinal is considered as the set of all smaller ordinals. A cardinal is identified with the least ordinal of this cardinality. By τ we denote a **fixed** infinite cardinal. By $\mathcal{F}$ we denote the set of all non-empty finite subsets of τ. The symbol $\equiv$ in a relation means that one or both sides of the relation are new notations. All spaces are assumed to be T_0-spaces of weight $\leq \tau$. An equivalent relation on a set X is considered as a subset of $X \times X$.

1.1 On the sheaves. We consider the notion of a sheaf according to [1]. A *sheaf of Abelian groups* is a triad $(\mathcal{A}, \pi, X)$ satisfying the following conditions:

(i) $\mathcal{A}$ and X are topological spaces and π is a map of $\mathcal{A}$ onto X.

(ii) π is a local homeomorphism, that is each point $a \in \mathcal{A}$ has an open neighbourhood V in $\mathcal{A}$ such that the restriction of π on V is a homeomorphism of V onto an open subset of X;

(iii) for each point $x \in X$ the set $\mathcal{A}_x \equiv \pi^{-1}(x)$, which is called *fiber of* $\mathcal{A}$ *in* x, is an Abelian group;

(iv) the group operations are continuous. (This condition means the following. Let $\mathcal{A} \boxtimes \mathcal{A}$ be the set of all pairs $(a, b) \in \mathcal{A} \times \mathcal{A}$ such that $\pi(a) = \pi(b)$. Then, the mapping $\varpi_{\mathcal{A}} : \mathcal{A} \boxtimes \mathcal{A} \to \mathcal{A}$ for which $\varpi_{\mathcal{A}}(a, b) = a + b$ is continuous. Similarly, the mapping $i_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ for which $i_{\mathcal{A}}(a) = -a$ is continuous.)

Below we give some well-known notions of sheaves and introduce some notations, which will be used in the paper. Let $p_1 \equiv (\mathcal{A}_1, \pi_1, X_1)$ and $p_2 \equiv (\mathcal{A}_2, \pi_2, X_2)$ be two sheaves. A continuous mapping f of $\mathcal{A}_1$ into $\mathcal{A}_2$ is called *homomorphism* if the restriction of f onto each fiber of $\mathcal{A}_1$ is a homomorphism of this fiber into a fiber of $\mathcal{A}_2$. The unique mapping g of X_1 into X_2 satisfying the relation $g \circ \pi_1 = \pi_2 \circ f$ is called *induced by* f. The homomorphism f is called *isomorphism* (or *embedding*) of p_1 into p_2 if f and the induced mapping g are embeddings. The isomorphism f of p_1 into p_2 is called *proper* if for each $x \in X_1$ the restriction of f onto the fiber $\mathcal{A}_{1,x}$ of $\mathcal{A}_1$ in x maps $\mathcal{A}_{1,x}$ **onto** the fiber $\mathcal{A}_{2,g(x)}$ of $\mathcal{A}_2$ in $g(x)$. The sheaves p_1 and p_2 are called *isomorphic* if there exists an isomorphism of p_1 onto p_2.

Let $(\mathcal{A}, \pi, X)$ be a sheaf and U a non-empty subset of X. A continuous mapping $s : U \to \mathcal{A}$, for which the maping $\pi \circ s$ is the identical mapping of U, is called *section of* $\mathcal{A}$ *on* U. We shall consider sections on the open subsets of X and the set of all such sections will be denoted by $\mathcal{A}(sec)$. The set of all sections on an open subset U of X will be denoted by $\mathcal{A}(sec)(U)$. The set $\mathcal{A}(sec)(U)$ is an Abelian group under the pointwise operations. For two subsets U and V such that $U \subset V$ we denote by $r^{\mathcal{A}}_{U,V}$ the mapping of $\mathcal{A}(sec)(V)$ into $\mathcal{A}(sec)(U)$ by setting $r^{\mathcal{A}}_{U,V}(s) = s|_U$ for each $s \in \mathcal{A}(sec)(V)$. The section $s|_U$ will be called *restriction* of the section s and the mapping $r^{\mathcal{A}}_{U,V}$ *restriction mapping*. Obviously, $r^{\mathcal{A}}_{U,V}$ is a homomorphism of the group $\mathcal{A}(sec)(V)$ into the group $\mathcal{A}(sec)(U)$. Moreover, if U, V, W are subsets of X and $U \subset V \subset W$, then $r^{\mathcal{A}}_{U,V} \circ r^{\mathcal{A}}_{V,W} = r^{\mathcal{A}}_{U,W}$.

For each section $s : U \to \mathcal{A}$ the set $dom(s) \equiv U$ is called the *domain of* s and the set $ran(s) \equiv s(U)$ is called the *range of* s. For each subset $B \subset \mathcal{A}(sec)$ and $x \in X$ we put

$$dom(B) = \{dom(s) : s \in B\}, \;\; ran(B) = \{ran(s) : s \in B\}$$

and

$$dom(B)(x) = \{dom(s) : s \in B, x \in dom(s)\}.$$

Also, for each non-empty subset $U \subset X$, we put

$$B(U) = \{s \in B : dom(s) = U\}.$$

We note that for each $s \in \mathcal{A}(sec)$ the set $ran(s)$ is an open subset of $\mathcal{A}$ and that the set $ran(\mathcal{A}(sec))$ is a base for the open subsets of $\mathcal{A}$.

The set $dom(\mathcal{A}(sec))(x)$ is directed by inclusion "$\subset$". Thus, for each $x \in X$ we have a direct spectrum of Abelian groups

$$\Sigma^{\mathcal{A}}_x \equiv \{\mathcal{A}(sec)(U), r^{\mathcal{A}}_{U,V}, dom(\mathcal{A}(sec))(x)\},$$

where $U, V \in dom(\mathcal{A}(sec))(x)$ with $U \subset V$ and $r^{\mathcal{A}}_{U,V}$ is the restriction mapping of cuts. The mapping

$$\vartheta^{\mathcal{A}}_x : \varinjlim \Sigma^{\mathcal{A}}_x \to \mathcal{A}_x,$$

of the limit group $\varinjlim \Sigma^{\mathcal{A}}_x$ of the spectrum $\sigma^{\mathcal{A}}_x$ into $\mathcal{A}_x$, defined by relation $\vartheta^{\mathcal{A}}_x(\sigma) = s(x)$, where σ is an arbitrary element of the limit group $\varinjlim \Sigma^{\mathcal{A}}_x$ and $s \in \sigma$, is an isomorphism of $\varinjlim \Sigma^{\mathcal{A}}_x$ onto $\mathcal{A}_x$.

1.2 On the Containing Spaces. In this section we briefly explain the construction of the Containing Spaces (see [7], [9]). The spaces of the universal sheaves in the main result of the paper (see below Theorem 1.3.1) will be Containing Spaces. A Containing Space is constructed for a given indexed collection $\mathbf{S}$ of spaces and it is uniquely determined by a *base* $\mathbf{B}$ *for* $\mathbf{S}$ (in [7] and [9] the base $\mathbf{B}$ is called mark and it is denoted by $\mathbf{M}$):

$$\mathbf{B} \equiv \{\{U^X_\delta : \delta \in \tau\} : X \in \mathbf{S}\},$$

where $\{U^X_\delta : \delta \in \tau\}$ is an indexed base for the open subsets of $X \in \mathbf{S}$, and by a *family* R *of equivalence relations on* $\mathbf{S}$:

$$\mathrm{R} \equiv \{\sim^t : t \in \mathcal{F}\}.$$

It is required that R satisfies the following conditions:

(a) for each $t \in \mathcal{F}$ the number of equivalence classes of $\sim^t$ is finite;

(b) for each $t_1 \subset t_2 \in \mathcal{F}$, $\sim^{t_2} \subset \sim^{t_1}$;

(c) for each $X, Y \in \mathbf{S}$ the condition $X \sim^t Y$ for some $t \in \mathcal{F}$, implies that the algebra of subsets of X, generated by the set $\{U^X_\delta : \delta \in t\}$, and the algebra of subsets of Y, generated by the set $\{U^Y_\delta : \delta \in t\}$, are isomorphic and the correspondence $U^X_\delta \to U^Y_\delta$, $\delta \in t$, generates an isomorphism of these algebras. Such a family is called $\mathbf{B}$-admissible. Also, for each $t \in \mathcal{F}$ we denote by $\mathrm{C}(\sim^t)$ the set of equivalence classes of $\sim^t$ and put $\mathrm{C(R)} = \cup\{\mathrm{C}(\sim^t) : t \in \mathcal{F}\}$. The corresponding Containing Space is denoted by $\mathrm{T} \equiv \mathrm{T}(\mathbf{B}, \mathrm{R})$ and its construction is done as follows.

Let $\sim^{\mathbf{B}}_{\mathrm{R}}$ be the equivalence relation on a set of all pairs (x, X), where $x \in X \in \mathbf{S}$, defined as follows: two such pairs (x, X) and (y, Y) are $\sim^{\mathbf{B}}_{\mathrm{R}}$-equivalent if and only if: (a) $X \sim^t Y$ for each $t \in \mathcal{F}$ and (b) for each $\delta \in \tau$, $x \in U^X_\delta$ if and only if $y \in U^Y_\delta$. Then, T is the set of all equivalence classes of $\sim^{\mathbf{B}}_{\mathrm{R}}$ and the set

$$B^{\mathrm{T}} \equiv \{U^{\mathrm{T}}_\delta(\mathbf{H}) : \delta \in \tau, \mathbf{H} \in \mathrm{C(R)}\},$$

where $U^{\mathrm{T}}_\delta(\mathbf{H})$ is the set consisting of all points $\mathbf{a} \in \mathrm{T}$ such that there exists an element $(x, X) \in \mathbf{a}$ for which $X \in \mathbf{H}$ and $x \in U^X_\delta$, is a base for a topology on T, called *standard base* (see Corollary 2.8 of [7]). We note that if for some $\kappa \subset \tau$ and for each $X \in \mathbf{S}$ the set $\{U^X_\delta : \delta \in \kappa\}$ is a base for the open subsets of X, then the set

$$\{U^{\mathrm{T}}_\delta(\mathbf{H}) : \delta \in \kappa, \mathbf{H} \in \mathrm{C(R)}\}$$

is also a base for the open subsets of the space T (see Corollary 2.8 of [7]).

The mapping $i^X_{\mathrm{T}} : X \to \mathrm{T}$, defining by the relation $i^X_{\mathrm{T}}(x) = \mathbf{a} \in \mathrm{T}$, where $x \in X \in \mathbf{S}$ and $\mathbf{a}$ is the point of T containing the pair (x, X), is an embedding of X into T, which is called *natural* (see Proposition 2.10 of [7]).

In the paper, we shall use also the following notions. Let

$$\mathbf{B}_1 \equiv \{\{U^X_{1,\delta} : \delta \in \tau\} : X \in \mathbf{S}\} \text{ and } \mathbf{B}_2 \equiv \{\{U^X_{2,\delta} : \delta \in \tau\} : X \in \mathbf{S}\},$$

where $\{U^X_{1,\delta} : \delta \in \tau\}$ and $\{U^X_{2,\delta} : \delta \in \tau\}$ are indexed sets of subsets of $X \in \mathbf{S}$ (in particular, they may be indexed bases of X) and $\mathbf{B}_2$ is a base for $\mathbf{S}$. The base $\mathbf{B}_2$ is an *extension* of $\mathbf{B}_1$ if there exists an one-to-one mapping $\vartheta : \tau \to \tau$, called *extension mapping*, such that $U^X_{1,\delta} = U^Y_{2,\vartheta(\delta)}$, $\delta \in \tau$. We shall also say that for a given $X \in \mathbf{S}$, $\{U^X_{2,\delta} : \delta \in \tau\}$ is an *extension* of $\{U^X_{1,\delta} : \delta \in \tau\}$ with the *extension mapping* ϑ. Let

$$\mathrm{R}_1 \equiv \{\sim^t_1 : t \in \mathcal{F}\}, \text{ and } \mathrm{R}_2 \equiv \{\sim^t_2 : t \in \mathcal{F}\}$$

be two families of equivalence relations on $\mathbf{S}$. We say that R_2 is a *final refinement* of R_1 if for each $t \in \mathcal{F}$ there exists $t' \in \mathcal{F}$ such that $\sim_2^{t'} \subset \sim_1^{t}$.

A class $\mathbb{S}$ of spaces is called *saturated* if for each indexed collection $\mathbf{S}$ of elements of $\mathbb{S}$, there exists a base $\mathbf{B}_0$ for $\mathbf{S}$ such that for each extension $\mathbf{B}$ of $\mathbf{B}_0$, there exists a $\mathbf{B}$-admissible family $\mathrm{R}^{\mathbf{B}}$ of equivalence relations on $\mathbf{S}$ with the property that for each admissible family R of equivalence relations on $\mathbf{S}$ being a final refinement of $\mathrm{R}^{\mathbf{B}}$, the containing space $\mathrm{T}(\mathbf{B}, \mathrm{R})$ belongs to $\mathbb{S}$ (see Section 3 of [7] and Chapter 2 of [9]). The base $\mathbf{B}_0$ is called *initial base* for $\mathbf{S}$ (*corresponding to the class* $\mathbb{S}$) and $\mathrm{R}^{\mathbf{B}}$ *initial family of equivalence relations on* $\mathbf{S}$ *corresponding to* $\mathbf{B}$ (*and the class* $\mathbb{S}$). Below, we give some examples of saturated classes of spaces of weight $\leq \tau$.

(1) the class of all T_0-spaces (see Propositions 2.9 of [7]);
(2) the class of all regular spaces (see Propositions 3.5 of [7]);
(3) the class of all completely regular spaces (see Propositions 3.8 of [7]);
(4) the class of all spaces of small inductive dimension $ind \leq n \in \mathbb{N}$;
(5) the class of all countable-dimensional spaces;
(6) the class of all strongly contable-dimensional spaces;
(7) the class of all locally finite-dimensional spaces;
(8) the intersection of any two saturated classes of spaces.

(For the above example (4) see Corrolary 3.1.6 of [9], for (5), (6) and (7) see Proposition 4.4.4 of [9] and for example (8) see Proposition 3.3 of [7]).

1.3 The results. Let $\mathbb{S}$ be a class of sheaves. A sheaf $\bar{p}$ is called *proper universal in the class* $\mathbb{S}$ if $\bar{p} \in \mathbb{S}$ and for each $p \in \mathbb{S}$ there exist a proper isomorphism of p into $\bar{p}$.

The main result of this paper is the following theorem.

Theorem 1.3.1. *Let $\mathbb{S}_d$ and $\mathbb{S}_r$ be two saturated classes of spaces of weights $\leq \tau$. Then, in the class of all sheaves $(\mathcal{A}, \pi, X)$, for which $\mathcal{A} \in \mathbb{S}_d$ and $X \in \mathbb{S}_r$, there exists a proper universal element $(\bar{\mathcal{A}}, \bar{\pi}, \bar{X})$.*

Since the class of T_0-spaces of countable weight and the class of separable metric spaces are saturated classes we have the following corollary.

Corollary 1.3.2. *In the class of all sheaves $(\mathcal{A}, \pi, X)$, where $\mathcal{A}$ is a T_0-space of countable weight and X is a separable metrizable space there exists a proper universal element.*

2. Proof of the result

Lemma 2.1. *Let $(\mathcal{A}, \pi, X)$ be a sheaf of Abelian groups. There exists a subset $B \subset \mathcal{A}(sec)$ such that:*

(*a*) *$ran(B)$ is a base for the open subsets of $\mathcal{A}$ of cardinality $w(\mathcal{A}) \leq \tau$ and, therefore, the set $dom(B)$ is a base for the open subsets of X;*

(*b*) *for each $U \in dom(B)$, $B(U)$ is a subgroup of the group $\mathcal{A}(sec)(U)$.*

(*c*) *for each* $U, V \in dom(B)$ *with* $U \subset V$ *the restriction* $s|_U$ *of any cut* $s \in B(V)$ *belongs to* B.

Proof. Since $\mathcal{A}(sec)$ is a base for $\mathcal{A}$, there exists a subset $B^0 \subset \mathcal{A}(sec)$ such that $ran(B^0)$ is a base for the open subsets of $\mathcal{A}$ of cardinality $w(\mathcal{A})$. By induction we define the subset $B^n \subset \mathcal{A}(sec)$, $n \in \mathbb{N}$, setting

$$B^n = B^{n-1} \cup (\cup\{B_+^{n-1}(U) \subset \mathcal{A}(sec)(U) : U \in dom(B^{n-1})\})$$
$$\cup\{s|_U : s \in B^{n-1}(V), U, V \in dom(B^{n-1}), U \subset V\},$$

where $B_+^{n-1}(U)$ is the subgroup of $\mathcal{A}(sec)(U)$ generated by the set $B^{n-1}(U)$. It is easy to see that the set $B \equiv \cup\{B^n : n \in \omega\}$ is the required set. □

The direct spectrum Σ_x^B. Let $(\mathcal{A}, \pi, X)$ be a sheaf, B a subset of $\mathcal{A}(sec)$, satisfying the conditions of Lemma 2.1, and $x \in X$. By property (a) of this lemma, $ran(B)$ is a base of $\mathcal{A}$ and, therefore, the set $dom(B)$ is a base for the open subsets of X. Hence, the set $dom(B)(x)$ is directed by inclusion "$\subset$". By property (c) of Lemma 2.1, for each $U, V \in dom(B)(x)$ with $U \subset V$ the restriction of $r_{U,V}^{\mathcal{A}}$ onto $B(V)$ is an isomorphism of $B(V)$ into $B(U)$. We shall denote this restriction by $r_{U,V}^B$. Thus, for each $x \in X$ we have a direct spectrum Σ_x^B of groups:

$$\Sigma_x^B \equiv \{B(U), r_{U,V}^B, dom(B)(x)\}. \tag{2.1.1}$$

Let σ^B be an arbitrary element of the limit group $\varinjlim \Sigma_x^B$ of the spectrum (2.1.1) and $s \in \sigma^B$. We define the mapping $\vartheta_x^B : \varinjlim \Sigma_x^B \to \mathcal{A}_x$ setting $\vartheta_x^B(\sigma^B) = s(x)$.

Lemma 2.2. *Let*

$$\vartheta_x^{B,\mathcal{A}} : \varinjlim \Sigma_x^B \to \varinjlim \Sigma_x^{\mathcal{A}}$$

be the mapping defined as follows: for each $\sigma^B \in \varinjlim \Sigma_x^B$ *we put* $\vartheta_x^{B,\mathcal{A}}(\sigma^B) = \sigma^{\mathcal{A}}$, *where* $\sigma^{\mathcal{A}}$ *is the element of* $\varinjlim \Sigma_x^{\mathcal{A}}$ *containing* σ^B. *Then,* $\vartheta_x^{B,\mathcal{A}}$ *is well-defined (that is, the element* $\sigma^{\mathcal{A}}$ *is uniquely determined), one-to-one, onto and preserves the group operations, that is it is an isomorphism of* $\varinjlim \Sigma_x^B$ *onto* $\varinjlim \Sigma_x^{\mathcal{A}}$. *Moreover,* $\vartheta_x^B = \vartheta_x^{\mathcal{A}} \circ \vartheta_x^{B,\mathcal{A}}$ *and, therefore,* ϑ_x^B *is an isomorphism and onto mapping.*

Proof. Since the mappings $r_{U,V}^B$ of the spectrum Σ_x^B are the restrictions of the corresponding mappings $r_{U,V}^{\mathcal{A}}$ of the spectrum $\Sigma_x^{\mathcal{A}}$ each element σ^B of $\varinjlim \Sigma_x^B$ is contained in an uniquely determined element $\sigma^{\mathcal{A}}$ of $\varinjlim \Sigma_x^{\mathcal{A}}$, that is the mapping $\vartheta_x^{B,\mathcal{A}}$ is well-defined.

We prove that $\vartheta_x^{B,\mathcal{A}}$ is one-to-one. Let σ_1^B and σ_2^B be two distinct elements of $\varinjlim \Sigma_x^B$ and let $\sigma_1^{\mathcal{A}}, \sigma_2^{\mathcal{A}} \in \varinjlim \Sigma_x^{\mathcal{A}}$ such that $\sigma_1^B \subset \sigma_1^{\mathcal{A}}$ and $\sigma_2^B \subset \sigma_2^{\mathcal{A}}$. Suppose that $\sigma_1^{\mathcal{A}} = \sigma_2^{\mathcal{A}}$ and let $s_1 \in \sigma_1^B$ and $s_2 \in \sigma_2^B$ and, therefore, $s_1, s_2 \in \sigma_1^{\mathcal{A}}$. Then, there exists $s_3 \in \sigma_1^{\mathcal{A}}$, which is a restriction of s_1 and a restriction of s_2. Since $dom(B)$ is a base for the open subsets of X (see property (a) of Lemma 2.1) there exists $s_0 \in B$ such that $x \in dom(s_0) \subset dom(s_3)$ and $ran(s_0) \subset ran(s_3)$.

Then, s_0 is the restriction of s_3 and, therefore, the restriction of s_1 and s_2, which contradicts the fact that s_1 and s_2 belong to distinct elements of $\varinjlim \Sigma_x^B$. Thus, $\vartheta_x^{B,\mathcal{A}}$ is one-to-one.

We prove that $\vartheta_x^{B,\mathcal{A}}$ is onto. Let $\sigma^{\mathcal{A}}$ be an element of $\varinjlim \Sigma_x^{\mathcal{A}}$ and $s \in \sigma_x^{\mathcal{A}}$. Consider an element $s_0 \in B$ such that $x \in dom(s_0) \subset dom(s)$ and $ran(s_0) \subset ran(s)$. Then for the element σ^B containing s_0 we have $\vartheta_x^{B,\mathcal{A}}(\sigma^B) = \sigma^{\mathcal{A}}$, proving that $\vartheta_x^{B,\mathcal{A}}$ is onto.

We prove that $\vartheta_x^{B,\mathcal{A}}$ preserves the group operations. Let $\sigma_1^B, \sigma_2^B \in \varinjlim \Sigma_x^B$ and let $\vartheta_x^{B,\mathcal{A}}(\sigma_1^B) = \sigma_1^{\mathcal{A}}$ and $\vartheta_x^{B,\mathcal{A}}(\sigma_2^B) = \sigma_2^{\mathcal{A}}$. Let $s_1 \in \sigma_1^B$ and $s_2 \in \sigma_2^B$. Consider an element $s_0 \in B$ such that $dom(s_0) \subset dom(s_1) \cap dom(s_2)$. Let $s_1' = s_1|_{dom(s_0)}$ and $s_2' = s_2|_{dom(s_0)}$. Then, by proprety (c) of Lemma 2.1, $s_1', s_2' \in B$ and by property (b) of this lemma, $s_1' + s_2' \in B$. Therefore, $s_1' + s_2' \in \sigma_1^B + \sigma_2^B$. On the other hand, $s_1' \in \sigma_1^{\mathcal{A}}$ and $s_2' \in \sigma_2^{\mathcal{A}}$ and, therefore, $s_1' + s_2' \in \sigma_1^{\mathcal{A}} + \sigma_2^{\mathcal{A}}$, proving that $\vartheta_x^{B,\mathcal{A}}$ preserves the sum operation. Similarly, we can prove that $\vartheta_x^{B,\mathcal{A}}$ preserves the taking of the inverse element. Thus, the mapping $\vartheta_x^{B,\mathcal{A}}$ is an isomorphism of $\varinjlim \Sigma_x^B$ onto $\varinjlim \Sigma_x^{\mathcal{A}}$.The relation $\vartheta_x^B = \vartheta_x^{\mathcal{A}} \circ \vartheta_x^{B,\mathcal{A}}$ is easy to verify. □

The indexed collections S, A and X. Consider the saturated classes $\mathbb{S}_d$ and $\mathbb{S}_r$ of the theorem. By set-theoretical reasons we can suppose that there exists a collection $\mathbf{S}$ of sheaves $(\mathcal{A}, \pi, X)$ such that $\mathcal{A} \in \mathbb{S}_d$, $X \in \mathbb{S}_r$ and each sheaf $(\mathcal{A}', \pi', X')$, for which $\mathcal{A}' \in \mathbb{S}_d$ and $X' \in \mathbb{S}_r$, is isomorphic to an element of $\mathbf{S}$. Moreover, we can suppose that $\mathbf{S}$ is indexed by a set Λ:

$$\mathbf{S} \equiv \{(\mathcal{A}_\lambda, \pi_\lambda, X_\lambda) : \lambda \in \Lambda\}.$$

We put

$$\mathbf{A} \equiv \{\mathcal{A}_\lambda : \lambda \in \Lambda\}, \;\; \mathbf{X} \equiv \{X_\lambda : \lambda \in \Lambda\}.$$

and consider $\mathbf{A}$ and $\mathbf{X}$ as indexed by Λ sets of topological spaces.

The bases $\mathbf{B}^{\mathbf{A}}$ and $\mathbf{B}^{\mathbf{X}}$ for A and X, respectively. For each element $(\mathcal{A}_\lambda, \pi_\lambda, X_\lambda) \in \mathbf{S}$ we consider a subset $B_\lambda \subset \mathcal{A}_\lambda(sec)$ satisfying the conditions of Lemma 2.1. Since $|B_\lambda| = w(\mathcal{A}_\lambda) \leq \tau$ (see the property (a) of Lemma 2.1), we can suppose that B_λ is indexed by the set τ:

$$B_\lambda = \{s_\eta^\lambda : \eta \in \tau\}.$$

Furthermore, we put

$$B_0^{\mathcal{A}_\lambda} \equiv \{V_\eta^{\mathcal{A}_\lambda} \equiv ran(s_\eta^\lambda) : \eta \in \tau\}, \;\text{ and }\; B_0^{X_\lambda} \equiv \{V_\eta^{X_\lambda} \equiv dom(s_\eta^\lambda) : \eta \in \tau\}.$$

Let θ_0 and θ_1 be two one-to-one mappings of τ into itself such that

$$|\theta_0(\tau)| = |\theta_1(\tau)|, \;\; \theta_0(\tau) \cap \theta_1(\tau) = \varnothing \;\text{ and }\; \theta_0(\tau) \cup \theta_1(\tau) = \tau.$$

(We note that these mappings are independed on $\lambda \in \Lambda$.) For each $\lambda \in \Lambda$ we put

$$W_{\zeta}^{\mathcal{A}_\lambda} = V_{\theta_0^{-1}(\zeta)}^{\mathcal{A}_\lambda} \text{ if } \zeta \in \theta_0(\tau) \text{ and } W_{\zeta}^{\mathcal{A}_\lambda} = \pi_\lambda^{-1}(V_{\theta_1^{-1}(\zeta)}^{X_\lambda}) \text{ if } \zeta \in \theta_1(\tau).$$

Therefore, the indexed set

$$B_1^{\mathcal{A}_\lambda} \equiv \{W_\zeta^{\mathcal{A}_\lambda} : \zeta \in \tau\}$$

is an extension of the indexed base $B_0^{\mathcal{A}_\lambda}$ of $\mathcal{A}_\lambda$ and, simultaneously, an extension of the indexed set

$$\pi_\lambda^{-1}(B_0^{X_\lambda}) \equiv \{\pi_\lambda^{-1}(V_\eta^{X_\lambda}) : \eta \in \tau\}$$

of subsets of $\mathcal{A}_\lambda$ with the extension mappings θ_0 and θ_1, respectively.

Now, we consider a base

$$\mathbf{B}^{\mathbf{A}} \equiv \{B^{\mathcal{A}_\lambda} \equiv \{U_\varepsilon^{\mathcal{A}_\lambda} : \varepsilon \in \tau\} : \lambda \in \Lambda\}$$

for $\mathbf{A}$, which is an initial base corresponding to the saturated class $\mathbb{S}_d$ and, simultaneously, is an extension of the base

$$\mathbf{B}_1^{\mathbf{A}} \equiv \{\{W_\zeta^{\mathcal{A}_\lambda} : \zeta \in \tau\} : \lambda \in \Lambda\}$$

for $\mathbf{A}$ with an extension mapping $\theta_{\mathbf{A}}$.

Also, we consider a base

$$\mathbf{B}^{\mathbf{X}} \equiv \{B^{X_\lambda} \equiv \{U_\delta^{X_\lambda} : \delta \in \tau\} : \lambda \in \Lambda\}$$

for $\mathbf{X}$, which is an initial base corresponding to the saturated class $\mathbb{S}_r$ and, simultaneously, is an extension of the base

$$\mathbf{B}_0^{\mathbf{X}} \equiv \{\{V_\eta^{X_\lambda} : \eta \in \tau\} : \lambda \in \Lambda\}$$

for $\mathbf{X}$ with an extension mapping $\theta_{\mathbf{X}}$.

The families $\mathrm{R}_{\mathbf{A}}$ and $\mathrm{R}_{\mathbf{X}}$ of equivalence relations. We denote by

$$\mathrm{R}_{\mathbf{A}} \equiv \{\sim_{\mathbf{A}}^t : t \in \mathcal{F}\}$$

a $\mathbf{B}^{\mathbf{A}}$-admissible family of equivalence relation on $\mathbf{A}$ and by

$$\mathrm{R}_{\mathbf{X}} \equiv \{\sim_{\mathbf{X}}^t : t \in \mathcal{F}\}$$

a $\mathbf{B}^{\mathbf{X}}$-admissible family of equivalence relations on $\mathbf{X}$. We suppose that $\mathrm{R}_{\mathbf{A}}$ and $\mathrm{R}_{\mathbf{X}}$ satisfy the following conditions:

(1) for each $\lambda, \mu \in \Lambda$ and $t \in \mathcal{F}$ the equivalence $X_\lambda \sim_{\mathbf{X}}^t X_\mu$ is true if and only if the equivalence $\mathcal{A}_\lambda \sim_{\mathbf{A}}^t \mathcal{A}_\mu$ is true;

(2) for each $\lambda, \mu \in \Lambda$, $t \in \mathcal{F}$, and $\eta_1, \eta_2, \eta \in t$ the equivalence $\mathcal{A}_\lambda \sim_{\mathbf{A}}^t \mathcal{A}_\mu$ implies that the conditions:

(2_1) $dom(s_{\eta_1}^\lambda) = dom(s_{\eta_2}^\lambda)$ and $s_{\eta_1}^\lambda + s_{\eta_2}^\lambda = s_\eta^\lambda$ and
(2_2) $dom(s_{\eta_1}^\mu) = dom(s_{\eta_2}^\mu)$ and $s_{\eta_1}^\mu + s_{\eta_2}^\mu = s_\eta^\mu$

are equivalent;

(3) for each $\lambda, \mu \in \Lambda$, $t \in \mathcal{F}$, and $\eta_1, \eta_2 \in t$ the equivalence $\mathcal{A}_\lambda \sim_{\mathbf{A}}^t \mathcal{A}_\mu$ implies that the conditions:

(3_1) $dom(s^{\lambda}_{\eta_1}) = dom(s^{\lambda}_{\eta_2})$ and $s^{\lambda}_{\eta_1} = -s^{\lambda}_{\eta_2}$ and
(3_2) $dom(s^{\mu}_{\eta_1}) = dom(s^{\mu}_{\eta_2})$ and $s^{\mu}_{\eta_1} = -s^{\mu}_{\eta_2}$
are equivalent ;

(4) for each $\lambda, \mu \in \Lambda$, $t \in \mathcal{F}$, and $\eta_1, \eta_2 \in t$ the equivalence $\mathcal{A}_\lambda \sim^t_{\mathbf{A}} \mathcal{A}_\mu$ implies that the conditions:
(4_1) $ran(s^{\lambda}_{\eta_1}) \subset ran(s^{\lambda}_{\eta_2})$ and
(4_2) $ran(s^{\mu}_{\eta_1}) \subset ran(s^{\mu}_{\eta_2})$
are equivalent;

(5) for each $\lambda, \mu \in \Lambda$, $t \in \mathcal{F}$, and $\eta_1, \eta_2 \in t$ the equivalence $\mathcal{A}_\lambda \sim^t_{\mathbf{A}} \mathcal{A}_\mu$ implies that the conditions:
(5_1) $dom(s^{\lambda}_{\eta_1}) \subset dom(s^{\lambda}_{\eta_2})$ and
(5_2) $dom(s^{\mu}_{\eta_1}) \subset dom(s^{\mu}_{\eta_2})$
are equivalent.

Lemma 2.3. *The* $\mathbf{B}^{\mathbf{A}}$*-admissible family* $\mathrm{R}_{\mathbf{A}}$ *and the* $\mathbf{B}^{\mathbf{X}}$*-admissible family* $\mathrm{R}_{\mathbf{X}}$ *satisfying conditions* $(1)-(5)$ *exist.*

Proof. Since the class $\mathbb{S}_d$ is saturated there exists a $\mathbf{B}^{\mathbf{A}}$-admissible family $\mathrm{R}_{0,\mathbf{A}} \equiv \{\sim^t_{0,\mathbf{A}}: t \in \mathcal{F}\}$, which is initial for the base $\mathbf{B}^{\mathbf{A}}$ and the class $\mathbb{S}_d$. Similarly, there exists $\mathbf{B}^{\mathbf{X}}$-admissible family $\mathrm{R}_{0,\mathbf{X}} \equiv \{\sim^t_{0,\mathbf{X}}: t \in \mathcal{F}\}$, which is initial for the base $\mathbf{B}^{\mathbf{X}}$ and the class $\mathbb{S}_r$.

Let $t \in \mathcal{F}$ and $\eta_1, \eta_2, \eta \in t$. We denote by $\sim^t_i$, $i \in \{2,3,4,5\}$, the equivalence relation on $\mathbf{A}$ defined as follows: $\mathcal{A}_\lambda \sim^t_i \mathcal{A}_\mu$, $\lambda, \mu \in \Lambda$, if and only if the conditions (i_1) and (i_2) are equivalent for all indexes η_1, η_2, η, which belong to t. Obviously, the relations $\sim^t_i$, $i \in \{2,3,4,5\}$, are admissible.

Let $\mathrm{R}_{1,\mathbf{A}} \equiv \{\sim^t_{1,\mathbf{A}}: t \in \mathcal{F}\}$ be the family of equivalence relations on $\mathbf{A}$, where $\sim^t_{1,\mathbf{A}} = \sim^t_{0,\mathbf{A}} \cap(\cap\{\sim^t_i : i \in \{2,3,4,5\}\})$ for each $t \in \mathcal{F}$. Now, for each $t \in \mathcal{F}$ we define the equivalence relation $\sim^t_{\mathbf{A}}$ on $\mathbf{A}$ as follows: $\mathcal{A}_\lambda \sim^t_{\mathbf{A}} \mathcal{A}_\mu$, $\lambda, \mu \in \Lambda$, if and only if $\mathcal{A}_\lambda \sim^t_{1,\mathbf{A}} \mathcal{A}_\mu$ and $X_\lambda \sim^t_{0,\mathbf{X}} X_\mu$. Also, we define the equivalence relation $\sim^t_{\mathbf{X}}$, $t \in \mathcal{F}$, on $\mathbf{X}$ as follows: $X_\lambda \sim^t_{\mathbf{X}} X_\mu$ if and only if $\mathcal{A}_\lambda \sim^t_{\mathbf{A}} \mathcal{A}_\mu$. It is easy to see that

$$\mathrm{R}_{\mathbf{A}} \equiv \{\sim^t_{\mathbf{A}}: t \in \mathcal{F}\} \text{ and } \mathrm{R}_{\mathbf{X}} \equiv \{\sim^t_{\mathbf{X}}: t \in \mathcal{F}\}$$

are the required families of equivalence relations. □

The equivalence relations $\sim_{\mathbf{A}}$ and $\sim_{\mathbf{X}}$. We put

$$\sim_{\mathbf{A}} = \cap\{\sim^t_{\mathbf{A}}: t \in \mathcal{F}\} \text{ and } \sim_{\mathbf{X}} = \cap\{\sim^t_{\mathbf{X}}: t \in \mathcal{F}\}.$$

The following two lemmas can easily be proved.

Lemma 2.4. *Let* $\mathcal{A}_\lambda \sim_{\mathbf{A}} \mathcal{A}_\mu$, $\lambda, \mu \in \Lambda$. *Then, the algebra of subsets of* $\mathcal{A}_\lambda$, *generated by the set* $B^{\mathcal{A}_\lambda}$, *and the algebra of subsets of* $\mathcal{A}_\mu$, *generated by the set* $B^{\mathcal{A}_\mu}$, *are isomorphic and the correspondence* $U^{\mathcal{A}_\lambda}_\varepsilon \to U^{\mathcal{A}_\mu}_\varepsilon$, $\varepsilon \in \tau$, *generates this isomorphism. Therefore, for any* $\kappa \subset \tau$ *the algebra of subsets of* $\mathcal{A}_\lambda$, *generated by the set* $\{U^{\mathcal{A}_\lambda}_\varepsilon : \varepsilon \in \kappa\}$, *and the algebra of subsets of* $\mathcal{A}_\mu$, *generated by the* *set $\{U_\varepsilon^{\mathcal{A}_\mu} : \varepsilon \in \kappa\}$, are isomorphic and the correspondence $U_\varepsilon^{\mathcal{A}_\lambda} \to U_\varepsilon^{\mathcal{A}_\mu}$, $\varepsilon \in \kappa$, generates this isomorphism. Moreover, for each $\eta, \eta_1, \eta_2 \in \tau$ we have:*

(a) the cut $s_{\eta_1}^\lambda$ is a restriction of the cut $s_{\eta_2}^\lambda$ if and only if the cut $s_{\eta_1}^\mu$ is the restriction of the cut $s_{\eta_2}^\mu$;

(b) the equalities $dom(s_{\eta_1}^\lambda) = dom(s_{\eta_2}^\lambda)$ and $s_\eta^\lambda = s_{\eta_1}^\lambda + s_{\eta_2}^\lambda$ are true if and only if the equalities $dom(s_{\eta_1}^\mu) = dom(s_{\eta_2}^\mu)$ and $s_\eta^\mu = s_{\eta_1}^\mu + s_{\eta_2}^\mu$ are true;

(c) the equalities $dom(s_{\eta_1}^\lambda) = dom(s_{\eta_2}^\lambda)$ and $s_{\eta_1}^\lambda = -s_{\eta_2}^\lambda$ are true if and only if the equalities $dom(s_{\eta_1}^\mu) = dom(s_{\eta_2}^\mu)$ and $s_{\eta_1}^\mu = -s_{\eta_2}^\mu$ are true.

Lemma 2.5. *Let $X_\lambda \sim_{\mathbf{X}} X_\mu$, $\lambda, \mu \in \Lambda$. Then, the algebra of subsets of X_λ, generated by the set B^{X_λ}, and the algebra of subsets of X_μ, generated by the set B^{X_μ}, are isomorphic and the correspondence $U_\delta^{X_\lambda} \to U_\delta^{X_\mu}$, $\delta \in \tau$, generates this isomorphism. Therefore, for any $k \subset \tau$ the algebra of subsets of X_λ, generated by the set $\{U_\delta^{X_\lambda} : \delta \in \kappa\}$, and the algebra of subsets of X_μ, generated by the set $\{U_\delta^{X_\mu} : \delta \in \kappa\}$, are isomorphic and the correspondence $U_\delta^{X_\lambda} \to U_\delta^{X_\mu}$, $\delta \in \kappa$, generates this isomorphism.*

The triad $(\bar{A}, \bar{\pi}, \bar{X})$. We put $\bar{A} = \mathrm{T}(\mathbf{B^A}, \mathrm{R}_\mathbf{A})$, $\bar{X} = \mathrm{T}(\mathbf{B^X}, \mathrm{R}_\mathbf{X})$ and define the mapping $\bar{\pi}$ as follows. Let $\mathbf{a} \in \bar{A}$ and $(a^\lambda, \mathcal{A}_\lambda) \in \mathbf{a}$ for some $\lambda \in \Lambda$. Then, we put $\bar{\pi}(\mathbf{a}) = \mathbf{x}$, where $\mathbf{x}$ is the point of $\bar{X}$ containing the pair $(\pi_\lambda(a^\lambda), X_\lambda)$. In what follows we shall prove that the triad $(\bar{A}, \bar{\pi}, \bar{X})$ is the required universal sheaf.

Lemma 2.6. *The mapping $\bar{\pi}$ is correctly defined (that is, it is independent from the element $(a^\lambda, \mathcal{A}_\lambda) \in \mathbf{a}$ considered in its definition).*

Proof. Let $\mathbf{a} \in \bar{A}$ and $(a^\lambda, \mathcal{A}_\lambda), (b^\mu, \mathcal{A}_\mu) \in \mathbf{a}$, that is $(a^\lambda, \mathcal{A}_\lambda)$ and $(b^\mu, \mathcal{A}_\mu)$ are $\sim_{\mathrm{R}_\mathbf{A}}^{\mathbf{B^A}}$-equivalent. We must prove that if $\pi_\lambda(a^\lambda) = x^\lambda$ and $\pi_\mu(b^\mu) = y^\mu$, then (x^λ, X_λ) and (y^μ, X_μ) are $\sim_{\mathrm{R}_\mathbf{X}}^{\mathbf{B^X}}$-equivalent, that is $X_\lambda \sim_\mathbf{X} X_\mu$ and for each $\delta \in \tau$ either $x^\lambda \in U_\delta^{X_\lambda}$ and $y^\mu \in U_\delta^{X_\mu}$ or $x^\lambda \notin U_\delta^{X_\lambda}$ and $y^\mu \notin U_\delta^{X_\mu}$. Since $(a^\lambda, \mathcal{A}_\lambda)$ and $(b_\mu, \mathcal{A}_\mu)$ are $\sim_{\mathrm{R}_\mathbf{A}}^{\mathbf{B^A}}$-equivalent, $\mathcal{A}_\lambda \sim_\mathbf{A} \mathcal{A}_\mu$. By the condition (1) of the definitions of $\mathrm{R}_\mathbf{A}$ and $\mathrm{R}_\mathbf{X}$ we have $X_\lambda \sim_\mathbf{X} X_\mu$. Suppose that there exists $\delta_0 \in \tau$ such that, for example, $x^\lambda \in U_{\delta_0}^{X_\lambda}$ and $y^\mu \notin U_{\delta_0}^{X_\mu}$. Then, $a^\lambda \in \pi_\lambda^{-1}(U_{\delta_0}^{X_\lambda})$ and $b^\mu \notin \pi_\mu^{-1}(U_{\delta_0}^{X_\mu})$.

Since the set $B_0^{X_\lambda}$ is a base for the open subsets of X_λ, there exists $\eta \in \tau$ such that $x^\lambda \in V_\eta^{X_\lambda} \subset U_{\delta_0}^{X_\lambda}$. Let $\delta_1 = \theta_\mathbf{X}(\eta)$ and, therefore, $V_\eta^{X_\lambda} = U_{\delta_1}^{X_\lambda}$ and $V_\eta^{X_\mu} = U_{\delta_1}^{X_\mu}$. Then, $U_{\delta_1}^{X_\lambda} \subset U_{\delta_0}^{X_\lambda}$. By Lemma 2.5 we have $U_{\delta_1}^{X_\mu} \subset U_{\delta_0}^{X_\mu}$ and, therefore, $y^\mu \notin U_{\delta_1}^{X_\mu} = V_\eta^{X_\mu}$. Since $\mathbf{B^A}$ is an extension of $\pi_\lambda^{-1}(B_0^{X_\lambda})$ with the extension mapping $\theta_\mathbf{A} \circ \theta_1$ we have $U_\varepsilon^{\mathcal{A}_\nu} = \pi_\nu^{-1}(V_\eta^{X_\nu})$ for each $\nu \in \Lambda$ and $\varepsilon = \theta_\mathbf{A}(\theta_1(\eta))$. Therefore, for λ and μ we have $U_\varepsilon^{\mathcal{A}_\lambda} = \pi_\lambda^{-1}(V_\eta^{X_\lambda})$ and $U_\varepsilon^{\mathcal{A}_\mu} = \pi_\mu^{-1}(V_\eta^{X_\mu})$, respectively, and hence $a^\lambda \in U_\varepsilon^{\mathcal{A}_\lambda}$ and $b^\mu \neq U_\varepsilon^{\mathcal{A}_\mu}$, which contradicts the fact that $(a^\lambda, \mathcal{A}_\lambda)$ and $(b^\mu, \mathcal{A}_\mu)$ are $\sim_{\mathrm{R}_\mathbf{A}}^{\mathbf{B^A}}$-equivalent. ☐

The following lemma can easily be verified.

Lemma 2.7. *For each $\lambda \in \Lambda$ the following relation is true:*

$$\bar{\pi} \circ i_{\bar{\mathcal{A}}}^{\mathcal{A}_\lambda} = i_{\bar{X}}^{X_\lambda} \circ \pi_\lambda, \tag{2.7.1}$$

where $i_{\bar{\mathcal{A}}}^{\mathcal{A}_\lambda}$ is the natural embedding of $\mathcal{A}_\lambda$ into $\bar{\mathcal{A}}$ and $i_{\bar{X}}^{X_\lambda}$ is the natural embedding of X_λ into $\bar{X}$.

Lemma 2.8. *For each $\eta \in \tau$ and $\mathbf{H} \in \mathrm{C}(\mathrm{R}_{\mathbf{A}})$ we have*

$$\bar{\pi}(U_\varepsilon^{\bar{\mathcal{A}}}(\mathbf{H})) = U_\delta^{\bar{X}}(\mathbf{L}),$$

where $\varepsilon = \theta_{\mathbf{A}}(\theta_0(\eta))$, $\delta = \theta_{\mathbf{X}}(\eta)$ and $\mathbf{L}$ is the element of $\mathrm{C}(\mathrm{R}_{\mathbf{X}})$, consisting of all elements $X_\lambda \in \mathbf{X}$ for which $\mathcal{A}_\lambda \in \mathbf{H}$ (we shall say that $\mathbf{L}$ and $\mathbf{H}$ correspond each other).

Proof. Let $\eta \in \tau$, $\varepsilon = \theta_{\mathbf{A}}(\theta_0(\eta))$, $\delta = \theta_{\mathbf{X}}(\eta)$ and $\mathbf{H} \in \mathrm{C}(\mathrm{R}_{\mathbf{A}})$. Then, for each $\lambda \in \Lambda$, $U_\varepsilon^{\mathcal{A}_\lambda} = im(s_\eta^\lambda)$. By the definition of the elements of the standard base of the Containing Spaces, we have

$$U_\varepsilon^{\bar{\mathcal{A}}}(\mathbf{H}) = \cup\{i_{\bar{\mathcal{A}}}^{\mathcal{A}_\lambda}(U_\varepsilon^{\mathcal{A}_\lambda}) : \mathcal{A}_\lambda \in \mathbf{H}\} = \cup\{i_{\bar{\mathcal{A}}}^{\mathcal{A}_\lambda}(ran(s_\eta^\lambda)) : \mathcal{A}_\lambda \in \mathbf{H}\}.$$

Therefore, using relation (2.7.1), we have

$$\bar{\pi}(U_\varepsilon^{\bar{\mathcal{A}}}(\mathbf{H})) = \cup\{\bar{\pi}(i_{\bar{\mathcal{A}}}^{\mathcal{A}_\lambda}(ran(s_\eta^\lambda))) : \mathcal{A}_\lambda \in \mathbf{H}\} =$$

$$\cup\{i_{\bar{X}}^{X_\lambda}(\pi_\lambda(ran(s_\eta^\lambda))) : X_\lambda \in \mathbf{L}\} = \cup\{i_{\bar{X}}^{X_\lambda}(dom(s_\eta^\lambda)) : X_\lambda \in \mathbf{L}\} =$$

$$\cup\{i_{\bar{X}}^{X_\lambda}(U_\delta^{X_\lambda})) : X_\lambda \in \mathbf{L}\} = U_\delta^{\bar{X}}(\mathbf{L}).$$

□

Proposition 2.9. *The mapping $\bar{\pi}$ is continuous.*

Proof. Since the set

$$\{U_\delta^{\bar{X}}(\mathbf{L}) : \delta \in \theta_{\mathbf{X}}(\tau), \mathbf{L} \in \mathrm{C}(\mathrm{R}_{\mathbf{X}})\}$$

is a base of the space $\bar{X}$ it suffices to prove that the set $\bar{\pi}^{-1}(\bar{U}_\delta^X(\mathbf{L}))$ is open in $\bar{\mathcal{A}}$ for each $\delta \in \theta_{\mathbf{X}}(\tau)$ and $\mathbf{L} \in \mathrm{C}(\mathrm{R}_{\mathbf{X}})$.

Let δ be a fixed element of $\theta_{\mathbf{X}}(\tau)$ and $\mathbf{L}$ a fixed element of $\mathrm{C}(\mathrm{R}_{\mathbf{X}})$. Let $\eta = \theta_{\mathbf{X}}^{-1}(\delta)$ and $\varepsilon = \theta_{\mathbf{A}}(\theta_1(\eta))$. Then, for each $\nu \in \Lambda$ we have

$$\pi_\nu^{-1}(U_\delta^{X_\nu}) = U_\varepsilon^{\mathcal{A}_\nu}. \tag{2.9.1}$$

We shall prove the following equality, which will prove the continuity of $\bar{\pi}$:

$$\bar{\pi}^{-1}(U_\delta^{\bar{X}}(\mathbf{L})) = U_\varepsilon^{\bar{\mathcal{A}}}(\mathbf{H}), \tag{2.9.2}$$

where $\mathbf{H}$ is the element of $\mathrm{C}(\mathrm{R}_{\mathbf{A}})$ corresponding to $\mathbf{L}$. Let $\mathbf{a} \in \bar{\pi}^{-1}(U_\delta^{\bar{X}}(\mathbf{L}))$, that is $\bar{\pi}(\mathbf{a}) \equiv \mathbf{x} \in U_\delta^{\bar{X}}(\mathbf{L})$. Let $(x^\lambda, X_\lambda) \in \mathbf{x}$ and $(a^\mu, \mathcal{A}_\mu) \in \mathbf{a}$. Since $\bar{\pi}(\mathbf{a}) = \mathbf{x}$, by the definition of $\bar{\pi}$ we have $(\pi_\mu(a^\mu), X_\mu) \in \mathbf{x}$. This means that $X_\lambda \sim_{\mathbf{X}} X\mu$, that is $X_\mu \in \mathbf{L}$ and, therefore, $\mathcal{A}_\mu \in \mathbf{H}$. Also, $\pi_\mu(a^\mu) \in U_\delta^{X_\mu}$ and, therefore,

$$a^\mu \in \pi_\mu^{-1}(U_\delta^{X_\mu}) = U_\varepsilon^{\mathcal{A}_\mu} \tag{2.9.3}$$

(see relation (2.9.1)). Since $\mathcal{A}_\mu \in \mathbf{H}$, the relation (2.9.3) shows that $\mathbf{a} \in U_\varepsilon^{\bar{\mathcal{A}}}(\mathbf{H})$, proving that the left side of the relation (2.9.2) is contained in the right.

Conversly, let $\mathbf{a} \in U_\varepsilon^{\bar{\mathcal{A}}}(\mathbf{H})$ and $(a^\mu, \mathcal{A}_\mu) \in \mathbf{a}$. Then, $\mathcal{A}_\mu \in \mathbf{H}$ and $a_\mu \in U_\varepsilon^{\mathcal{A}_\mu}$. Therefore, $X_\mu \in \mathbf{L}$ and $a^\mu \in \pi_\mu^{-1}(U_\delta^{X_\mu})$ (see relation 2.9.1) or $\pi_\mu(a^\mu) \in U_\delta^{X_\mu}$. This means that $\bar{\pi}(\mathbf{a}) \in U_\delta^{\bar{X}}(\mathbf{L})$ and, therefore, $\mathbf{a} \in \bar{\pi}^{-1}(U_\delta^{\bar{X}}(\mathbf{L}))$, proving that the right side of (2.9.2) is contained in the left, completing the proof of the proposition. □

The set $\bar{\mathcal{A}}_{\mathbf{x}}$, $\mathbf{x} \in \bar{X}$. For each $\mathbf{x} \in \bar{X}$ we put

$$\bar{\mathcal{A}}_{\mathbf{x}} = \{\mathbf{a} \in \bar{\mathcal{A}} : \bar{\pi}(\mathbf{a}) = \mathbf{x}\}.$$

We shall prove that $\bar{\mathcal{A}}_{\mathbf{x}}$ is an Abelian group. First we shall prove the following lemma.

Lemma 2.10. *Let $(\mathcal{A}_\lambda, \pi_\lambda, X_\lambda)$ and $(\mathcal{A}_\mu, \pi_\mu, X_\mu)$ be two elements of $\mathbf{S}$ such that $\mathcal{A}_\lambda \sim_{\mathbf{A}} \mathcal{A}_\mu$ and let $\mathbf{x} \in \bar{X}$. Then, for each two elements*

$$(x^\lambda, X_\lambda), (x^\mu, X_\mu) \in \mathbf{x} \in \bar{X}$$

there exists an isomorphism $\vartheta_{x_\mu}^{x_\lambda}$ of $\mathcal{A}_{\lambda, x^\lambda}$ onto $\mathcal{A}_{\mu, x^\mu}$ such that:
(a) $\vartheta_{x_\lambda}^{x_\lambda}$ is the identical isomorphism;
(b) $\vartheta_{x_\nu}^{x_\mu} \circ \vartheta_{x_\mu}^{x_\lambda} = \vartheta_{x_\nu}^{x_\lambda}$, where $(x^\nu, X_\nu) \in \mathbf{x}$.
Moreover, for each $a^\lambda \in \mathcal{A}_{\lambda, x^\lambda}$, we have

$$(a^\lambda, \mathcal{A}_\lambda) \sim_{\mathrm{R}_{\mathbf{A}}}^{\mathbf{B}^{\mathbf{A}}} (\vartheta_{x^\mu}^{x^\lambda}(a^\lambda), \mathcal{A}_\mu). \tag{2.10.1}$$

Proof. Let $(x^\lambda, X_\lambda), (x^\mu, X_\mu) \in \mathbf{x} \in \bar{X}$ for some fixed $\lambda, \mu \in \Lambda$. Then, $x^\lambda \in U_\delta^{X_\lambda}$ for some $\delta \in \theta_{\mathbf{X}}(\tau)$ if and only if $x^\mu \in U_\delta^{X_\mu}$, that is $x^\lambda \in dom(s_\eta^\lambda)$ for some $\eta \in \tau$, if and only if $x^\mu \in dom(s_\eta^\mu)$. Denote by κ all such η. By Lemma 2.5 it follows that the mapping $dom(s_\eta^\lambda) \to dom(s_\eta^\mu)$, $\eta \in \kappa$, is an isomorphism of the directed by inclusion set $dom(B_\lambda)(x^\lambda)$ onto the directed by inclusion set $dom(B_\mu)(x^\mu)$. Let

$$\sigma^\lambda \equiv \{s_\eta^\lambda : \eta \in \tau(\sigma^\lambda)\}$$

be an element of the limit group $\varinjlim \Sigma_{x^\lambda}^{B_\lambda}$, where $\tau(\sigma^\lambda)$ is the set of all $\eta \in \tau$ for which $s_\eta^\lambda \in \sigma^\lambda$. Consider the set

$$\sigma^\mu(\lambda) \equiv \{s_\eta^\mu : \eta \in \tau(\sigma^\lambda)\}$$

of sections of $\mathcal{A}_\mu$. Let $s_{\eta_1}^\lambda, s_{\eta_2}^\lambda \in \sigma^\lambda$ such that $ran(s_{\eta_1}^\lambda) \subset ran(s_{\eta_2}^\lambda)$, that is the section $s_{\eta_1}^\lambda$ is the restriction of the section $s_{\eta_2}^\lambda$. By relation $\mathcal{A}_\lambda \sim_{\mathbf{A}} \mathcal{A}_\mu$ and Lemma 2.4 it follows that $ran(s_{\eta_1}^\mu) \subset ran(s_{\eta_2}^\mu)$, that is the section $s_{\eta_1}^\mu$ is the restriction of the section $s_{\eta_2}^\mu$. This means that the set $\sigma^\mu(\lambda)$ is a subset of an unique determined element σ^μ of the limit group $\varinjlim \Sigma_{x^\mu}^{B_\mu}$.

Similarly, the constructed element σ^μ of the limit group $\varinjlim \Sigma_{x^\mu}^{B_\mu}$ defines a set $\sigma^\lambda(\mu)$ of sections of $\mathcal{A}_\lambda$. By construction, $\sigma^\lambda \subset \sigma^\lambda(\mu)$ and, therefore,

$\sigma^\lambda = \sigma^\lambda(\mu)$. Similarly, $\sigma^\mu = \sigma^\mu(\lambda)$. We will say that $\sigma^\lambda \in \varinjlim \Sigma^{B_\lambda}_{x^\lambda}$ and $\sigma^\mu \in \varinjlim \Sigma^{B_\mu}_{x^\mu}$ correspond each other. Thus, we have defined mutually inverse, one-to-one and onto mappings

$$\vartheta(x^\lambda, x^\mu) : \varinjlim \Sigma^{B_\lambda}_{x^\lambda} \to \varinjlim \Sigma^{B_\mu}_{x^\mu} \text{ and } \vartheta(x^\mu, x^\lambda) : \varinjlim \Sigma^{B_\mu}_{x^\mu} \to \varinjlim \Sigma^{B_\lambda}_{x^\lambda}.$$

We prove that the mapping $\vartheta(x^\lambda, x^\mu)$ is an isomorphism, that is it preserves the group operations. Let $\sigma^\lambda_1, \sigma^\lambda_2 \in \varinjlim \Sigma^{B_\lambda}_{x^\lambda}$ and let $s^\lambda_{\eta_1} \in \sigma^\lambda_1$ and $s^\lambda_{\eta_2} \in \sigma^\lambda_2$. Since the set $dom(B_\lambda)(x^\lambda)$ is a base of X^λ at the point x^λ, there exists $s^\lambda_\eta \in B_\lambda$ such that $U \equiv dom(s^\lambda_\eta) \subset dom(s^\lambda_{\eta_1}) \cap dom(s^\lambda_{\eta_2})$. By condition (c) of Lemma 2.1 the restrictions $s^\lambda_{\eta_1}|_U$ and $s^\lambda_{\eta_2}|_U$ belong to B_λ and by condition (b), $s^\lambda_{\eta_1}|_U + s^\lambda_{\eta_2}|_U$ belongs to B_λ. Then, there exist $\eta'_1, \eta'_2, \eta'_0 \in \tau$ such that

$$s^\lambda_{\eta'_1} = s^\lambda_{\eta_1}|_U \in \sigma^\lambda_1, \;\; s^\lambda_{\eta'_2} = s^\lambda_{\eta_2}|_U \in \sigma^\lambda_2 \;\text{ and }\; s^\lambda_{\eta'_0} = s^\lambda_{\eta'_1} + s^\lambda_{\eta'_2} \in \sigma^\lambda_1 + \sigma^\lambda_2.$$

Let

$$\vartheta(x^\lambda, x^\mu)(\sigma^\lambda_1) = \sigma^\mu_1, \;\; \vartheta(x^\lambda, x^\mu)(\sigma^\lambda_2) = \sigma^\mu_2, \;\text{ and }\; \vartheta(x^\lambda, x^\mu)(\sigma^\lambda) = \sigma^\mu.$$

By Lemmas 2.4 and 2.5 we have

$$V \equiv dom(s^\mu_\eta) \subset dom(s^\mu_{\eta_1}) \cap dom(s^\mu_{\eta 2}),$$

$$s^\mu_{\eta'_1} = s^\mu_{\eta_1}|_V \in \sigma^\mu_1, \;\; s^\mu_{\eta'_2} = s^\mu_{\eta_2}|_V \in \sigma^\mu_2, \;\text{ and, }\; s^\mu_{\eta'_0} = s^\mu_{\eta'_1} + s^\mu_{\eta'_2} \in \sigma^\mu_1 + \sigma^\mu_2$$

Therefore,

$$\vartheta(x^\lambda, x^\mu)(\sigma^\lambda_1 + \sigma^\lambda_2) = \vartheta(x^\lambda, x^\mu)(\sigma^\lambda_1) + \vartheta(x^\lambda, x^\mu)(\sigma^\lambda_2).$$

Similarly, we prove that $\vartheta(x^\lambda, x^\mu)(-\sigma^\lambda_1) = -\vartheta(x^\lambda, x^\mu)(\sigma^\lambda_1)$. Thus, the mapping $\vartheta(x^\lambda, x^\mu)$ and, therefore, the mapping $\vartheta(x^\mu, x^\lambda)$ is an isomorphism and onto.

The required isomorphism $\vartheta^{x^\lambda}_{x^\mu}$ of $\mathcal{A}_{\lambda, x^\lambda}$ onto $\mathcal{A}_{\mu, x^\mu}$ is defined by setting

$$\vartheta^{x^\lambda}_{x^\mu} = \vartheta^{B_\mu}_{x^\mu} \circ \vartheta(x^\lambda, x^\mu) \circ (\vartheta^{B_\lambda}_{x^\lambda})^{-1}.$$

Conditions (a) and (b) of the lemma can easily be verified.

We prove relation (2.10.1). Let $a^\lambda \in \mathcal{A}_{\lambda, x^\lambda}$ and $\sigma^\lambda_0 = (\vartheta^{B_\lambda}_{x^\lambda})^{-1}(a^\lambda)$. Then,

$$\vartheta^{x^\lambda}_{x^\mu}(a^\lambda) = \vartheta^{B_\mu}_{x^\mu}(\vartheta(x^\lambda, x^\mu)(\sigma^\lambda_0)) = \vartheta^{B_\mu}_{x^\mu}(\sigma^\mu_0),$$

where σ^μ_0 is the element $\varinjlim \Sigma^{B_\mu}_{x^\mu}$ corresponding to σ^λ_0. Thus, it suffices to prove that the pairs $(\vartheta^{B_\lambda}_{x^\lambda}(\sigma^\lambda_0), \mathcal{A}_\lambda)$ and $(\vartheta^{B^\mu}_{x^\mu}(\sigma^\mu_0), \mathcal{A}_\mu)$ belong to the same element of $\bar{\mathcal{A}}$. Let $U^{\mathcal{A}_\lambda}_\varepsilon$ be an element of $B^{\mathcal{A}_\lambda}$ containing $\vartheta^{B_\lambda}_{x^\lambda}(\sigma^\lambda)$. We need to prove that

$$\vartheta^{B_\mu}_{x^\mu}(\sigma^\mu) \in U^{\mathcal{A}_\mu}_\varepsilon. \tag{2.10.2}$$

Since the set $\{ran(s) : s \in \sigma^\lambda\}$ is a base for the open subsets of $\mathcal{A}_\lambda$ at the point $\vartheta^{B_\mu}_{x^\mu}(\sigma^\mu)$, there exists $\eta \in \tau$ such that $s^\lambda_\eta \in \sigma^\lambda$ and

$$\vartheta^{B_\lambda}_{x^\lambda}(\sigma^\lambda) \in ran(s^\lambda_\eta) \subset U^{\mathcal{A}_\lambda}_\varepsilon.$$

Since σ^μ corresponds to σ^λ, $s^\mu_\eta \in \sigma$. By the definition of the mapping $\vartheta^{B_\mu}_{x^\mu}$, $\vartheta^{B_\mu}_{x^\mu}(\sigma^\mu) = s^\mu_\eta(x^\mu) \in ran(s^\mu_\eta)$. Since $\mathcal{A}_\lambda \sim_{\mathbf{A}} \mathcal{A}_\mu$, the condition $ran(s^\lambda_\eta) \subset U^{\mathcal{A}_\lambda}_\varepsilon$ implies that $ran(s^\mu_\eta) \subset U^{\mathcal{A}_\mu}_\varepsilon$, proving relation (2.10.2). □

Proposition 2.11. *For each $\mathbf{x} \in \bar{X}$ the set $\bar{\mathcal{A}}_{\mathbf{x}}$ is an Abelian group and for each $(x^\lambda, X_\lambda) \in \mathbf{x}$, $\lambda \in \Lambda$, the natural embedding $i^{\mathcal{A}_\lambda}_{\bar{\mathcal{A}}}$ of $\mathcal{A}_\lambda$ into $\bar{\mathcal{A}}$ maps the fiber $\mathcal{A}_{\lambda,x^\lambda}$ of $\mathcal{A}_\lambda$* **onto** *the set $\bar{\mathcal{A}}_{\mathbf{x}}$.*

Proof. Let $\mathbf{x} \in \bar{X}$, $(x^\lambda, X_\lambda) \in \mathbf{x}$ for some fixed $\lambda \in \Lambda$ and $\mathbf{a} \in \bar{\mathcal{A}}_{\mathbf{x}}$. By relation (2.7.1) it follows that $i^{\mathcal{A}_\lambda}_{\bar{\mathcal{A}}}(\mathcal{A}_{\lambda,x^\lambda}) \subset \bar{\mathcal{A}}_{\mathbf{x}}$. We must prove that $i^{\mathcal{A}_\lambda}_{\bar{\mathcal{A}}}(\mathcal{A}_{\lambda,x^\lambda}) = \bar{\mathcal{A}}_{\mathbf{x}}$. By the definition of the Containing Spaces there exists $\mu \in \Lambda$ and a point $a^\mu \in \mathcal{A}_\mu$ such that $(a^\mu, \mathcal{A}_\mu) \in \mathbf{a}$. Using relation (2.7.1) we can see that $(x^\mu, X_\mu) \in \mathbf{x}$, where $x^\mu = \pi_\mu(a^\mu)$. Let $a^\lambda = (\vartheta^{x^\lambda}_{x^\mu})^{-1}(a_\mu)$. Then, by the relation (2.10.1), $(a^\lambda, \mathcal{A}_\lambda) \sim^{B^{\mathbf{A}}}_{\mathrm{R}_{\mathbf{A}}} (a^\mu, \mathcal{A}_\mu)$ and, therefore, $(a^\lambda, \mathcal{A}_\lambda) \in \mathbf{a}$, that is $i^{\mathcal{A}_\lambda}_{\bar{\mathcal{A}}}(a^\lambda) = \mathbf{a}$, proving that $i^{\mathcal{A}_\lambda}_{\bar{\mathcal{A}}}$ maps the fiber $\mathcal{A}_{\lambda,x^\lambda}$ of $\mathcal{A}_\lambda$ **onto** the set $\bar{\mathcal{A}}_{\mathbf{x}}$.

Now, on the set $\bar{\mathcal{A}}_{\mathbf{x}}$, $\mathbf{x} \in \bar{X}$, we define the group operations. Let $\mathbf{a}_1, \mathbf{a}_2 \in \bar{\mathcal{A}}_{\mathbf{x}}$ and let $(a^\lambda_1, \mathcal{A}_\lambda) \in \mathbf{a}_1$ and $(a^\lambda_2, \mathcal{A}_\lambda) \in \mathbf{a}_2$. Then, we put $\mathbf{a}_1 + \mathbf{a}_2 = \mathbf{a}$, where $\mathbf{a}$ is the element of $\bar{\mathcal{A}}_{\mathbf{x}}$, containing the pair $(a^\lambda_1 + a^\lambda_2, \mathcal{A}_\lambda)$. Also, we consider that $-\mathbf{a}_1$ is the element of $\bar{\mathcal{A}}_{\mathbf{x}}$, containing the pair $(-a^\lambda_1, \mathcal{A}_\lambda)$. Obviously, by these operations $\bar{\mathcal{A}}_{\mathbf{x}}$ becomes an Abelian group such that the restriction onto $\mathcal{A}_{\lambda,x^\lambda}$ of the natural embedding $i^{\mathcal{A}_\lambda}_{\bar{\mathcal{A}}}$ of $\mathcal{A}_\lambda$ into $\bar{\mathcal{A}}$, is an isomorphism of $\mathcal{A}_{\lambda,x^\lambda}$ onto $\bar{\mathcal{A}}_{\mathbf{x}}$.

It remains to prove that the defined operations are independent of the element $(x^\lambda, X_\lambda) \in \mathbf{x}$. Let $(x^\nu, X_\nu) \in \mathbf{x}$ for some $\nu \in \Lambda$, $(a^\nu_1, \mathcal{A}_\nu) \in \mathbf{a}_1$ and $(a^\nu_2, \mathcal{A}_\nu) \in \mathbf{a}_2$. We need to prove that the pair $(a^\lambda_1 + a^\lambda_2, \mathcal{A}_\lambda)$ and $(a^\nu_1 + a^\nu_2, \mathcal{A}_\nu)$ belong to the same element of $\bar{\mathcal{A}}_{\mathbf{x}}$. Since, by Lemma 2.10, $\vartheta^{x^\lambda}_{x^\nu}$ is an isomorphism we have

$$\vartheta^{x^\lambda}_{x^\nu}(a^\lambda_1 + a^\lambda_2) = \vartheta^{x^\lambda}_{x^\nu}(a^\lambda_1) + \vartheta^{x^\lambda}_{x^\nu}(a^\lambda_2).$$

On the other hand, since the pairs $(a^\lambda_1, \mathcal{A}_\lambda)$ and $(\vartheta^{x^\lambda}_{x^\nu}(a^\lambda_1), \mathcal{A}_\nu)$ belong to the same element of $\bar{\mathcal{A}}$ we have $(\vartheta^{x^\lambda}_{x^\nu}(a^\lambda_1), \mathcal{A}_\nu) \in \mathbf{a}_1$ and since $(a^\nu_1, \mathcal{A}_\nu) \in \mathbf{a}_1$ we have $\vartheta^{x^\lambda}_{x^\nu}(a^\lambda_1) = a^\nu_1$. Similarly, $\vartheta^{x^\lambda}_{x^\nu}(a^\lambda_2) = a^\nu_2$. Therefore, $\vartheta^{x^\lambda}_{x^\nu}(a^\lambda_1 + a^\lambda_2) = a^\nu_1 + a^\nu_2$. Since, by Lemma 2.10, the pairs $(a^\lambda_1 + a^\lambda_2, \mathcal{A}_\lambda)$ and $(\vartheta^{x^\lambda}_{x^\nu}(a^\lambda_1 + a^\lambda_2), \mathcal{A}_\nu)$ belong to the same element of $\bar{\mathcal{A}}$, the pairs $(a^\lambda_1 + a^\lambda_2, \mathcal{A}_\lambda)$ and $(a^\nu_1 + a^\nu_2, \mathcal{A}_\nu)$ also belong to the same element of $\bar{\mathcal{A}}$, proving that the sum operation is independent of the element $(x^\lambda, X_\lambda) \in \mathbf{x}$. Similarly, we prove that the operation of taking the inverse element is independent of $(x^\lambda, X_\lambda) \in \mathbf{x}$. The proof of the proposition is completed. □

The mappings $\varpi_{\bar{\mathcal{A}}}$ and $i_{\bar{\mathcal{A}}}$. We put

$$\bar{\mathcal{A}} \boxtimes \bar{\mathcal{A}} \equiv \{(\mathbf{a}, \mathbf{b}) \in \bar{\mathcal{A}} \times \bar{\mathcal{A}} : \bar{\pi}(\mathbf{a}) = \bar{\pi}(\mathbf{b})\}$$

and define the mappings

$$\varpi_{\bar{\mathcal{A}}} : \bar{\mathcal{A}} \boxtimes \bar{\mathcal{A}} \to \bar{\mathcal{A}} \text{ and } i_{\bar{\mathcal{A}}} : \bar{\mathcal{A}} \to \bar{\mathcal{A}}$$

setting

$$\varpi_{\bar{\mathcal{A}}}(\mathbf{a}, \mathbf{b}) = \mathbf{a} + \mathbf{b} \text{ and } i_{\bar{\mathcal{A}}}(\mathbf{a}) = -\mathbf{a}.$$

Proposition 2.12. *The mappings $\varpi_{\bar{\mathcal{A}}}$ and $i_{\bar{\mathcal{A}}}$ are continuous.*

Proof. We prove that $\varpi_{\bar{\mathcal{A}}}$ is continuous. Let $(\mathbf{a}_1, \mathbf{a}_2) \in \bar{\mathcal{A}} \boxtimes \bar{\mathcal{A}}$ and $\mathbf{a} \equiv \mathbf{a}_1 + \mathbf{a}_2$. Let U be an open neighbourhood of $\mathbf{a}$ in $\bar{\mathcal{A}}$. We must find open neighbourhoods U_1 and U_2 of $\mathbf{a}_1$ and $\mathbf{a}_2$, respectively, in $\bar{\mathcal{A}}$ such that

$$\varpi_{\bar{\mathcal{A}}}((U_1 \times U_2) \cap (\bar{\mathcal{A}} \boxtimes \bar{\mathcal{A}})) \subset U. \tag{2.12.1}$$

Without loss of generality, we can suppose that U is an element $U_\varepsilon^{\bar{\mathcal{A}}}(\mathbf{H})$ of the standard base of $\bar{\mathcal{A}}$, where ε is a fixed element of τ and $\mathbf{H}$ is a fixed element of $\mathrm{C}(\sim_{\mathbf{A}}^t)$ for some fixed $t \in \mathcal{F}$. Moreover, we can suppose that $\varepsilon \in \theta_{\mathbf{A}}(\theta_0(\tau))$. This means that for each $\mathcal{A}_\mu \in \mathbf{H}$ we have $U_\varepsilon^{\mathcal{A}_\mu} = ran(s_\eta^\mu)$, where $\eta = (\theta_{\mathbf{A}} \circ \theta_0)^{-1}(\varepsilon)$ and, therefore,

$$U_\varepsilon^{\bar{\mathcal{A}}}(\mathbf{H}) = \cup\{i_{\bar{\mathcal{A}}}^{\mathcal{A}_\mu}(U_\varepsilon^{\mathcal{A}_\mu}) : \mathcal{A}_\mu \in \mathbf{H}\} = \cup\{i_{\bar{\mathcal{A}}}^{\mathcal{A}_\mu}(ran(s_\eta^\mu)) : \mathcal{A}_\mu \in \mathbf{H}\}.$$

Let λ be a fixed element of Λ such that $\mathcal{A}_\lambda \in \mathbf{H}$ and $\mathbf{a} \in i_{\bar{\mathcal{A}}}^{\mathcal{A}_\lambda}(ran(s_\eta^\lambda))$. Therefore, there exists a point $a^\lambda \in ran(s_\eta^\lambda) \subset \mathcal{A}_\lambda$ such that $i_{\bar{\mathcal{A}}}^{\mathcal{A}_\lambda}(a^\lambda) = \mathbf{a}$, that is $(a^\lambda, \mathcal{A}_\lambda) \in \mathbf{a}$. Let $\pi_\lambda(a^\lambda) = x^\lambda \in X_\lambda$. Then, by relation (2.7.1), $i_{\bar{X}}^{X_\lambda}(x^\lambda) = \bar{\pi}(\mathbf{a}) \equiv \mathbf{x} \in \bar{X}$ and, therefore, $(x^\lambda, X_\lambda) \in \mathbf{x}$. By Lemma 2.10, there are points $a_1^\lambda, a_2^\lambda \in \mathcal{A}_{\lambda, x^\lambda}$ such that $(a_1^\lambda, \mathcal{A}_\lambda) \in \mathbf{a}_1$, $(a_2^\lambda, \mathcal{A}_\lambda) \in \mathbf{a}_2$ and $(a_1^\lambda + a_2^\lambda, \mathcal{A}_\lambda) \in \mathbf{a}$. Since $(a^\lambda, \mathcal{A}_\lambda) \in \mathbf{a}$ we have $a_1^\lambda + a_2^\lambda = a^\lambda \in ran(s_\eta^\lambda)$.

Since the mapping ϖ_λ is continuous, there exist $\varepsilon_1, \varepsilon_2 \in \tau$ such that $a_1^\lambda \in U_{\varepsilon_1}^{\mathcal{A}_\lambda}$, $a_2^\lambda \in U_{\varepsilon_2}^{\mathcal{A}_\lambda}$ and

$$\varpi_\lambda((U_{\varepsilon_1}^{\mathcal{A}_\lambda} \times U_{\varepsilon_2}^{\mathcal{A}_\lambda}) \cap (\mathcal{A}_\lambda \boxtimes \mathcal{A}_\lambda)) \subset ran(s_\eta^\lambda). \tag{2.12.2}$$

Without loss of generality, we can suppose that $\varepsilon_1, \varepsilon_2 \in \theta_{\mathbf{A}}(\theta_0(\tau))$, that is there exist cuts $s_{\eta_1}^\lambda$ and $s_{\eta_2}^\lambda$ such that $ran(s_{\eta_1}^\lambda) = U_{\varepsilon_1}^{\mathcal{A}_\lambda}$, $ran(s_{\eta_2}^\lambda) = U_{\varepsilon_2}^{\mathcal{A}_\lambda}$, $dom(s_{\eta_1}^\lambda) = dom(s_{\eta_2}^\lambda) \subset dom(s_\eta^\lambda)$. Condition (b) of Lemma 2.1 implies that there exists $\eta_0 \in \tau$ such that $s_{\eta_0}^\lambda = s_{\eta_1}^\lambda + s_{\eta_2}^\lambda$ and, therefore, $dom(s_{\eta_0}^\lambda) = dom(s_{\eta_i}^\lambda)$, $i = 1, 2$. In this case, the left side of the relation (2.12.2) takes the form

$$\varpi_\lambda((ran(s_{\eta_1}^\lambda) \times ran(s_{\eta_2}^\lambda)) \cap (\mathcal{A}_\lambda \boxtimes \mathcal{A}_\lambda)). \tag{2.12.3}$$

Since

$$\mathcal{A}_\lambda \boxtimes \mathcal{A}_\lambda = \cup\{\mathcal{A}_{\lambda, x^\lambda} \times \mathcal{A}_{\lambda, x^\lambda} : x^\lambda \in X_\lambda\},$$

the expression (2.12.3) takes the form

$$\varpi_\lambda(\cup\{(ran(s_{\eta_1}^\lambda) \cap \mathcal{A}_{\lambda, x^\lambda}) \times (ran(s_{\eta_2}^\lambda) \cap \mathcal{A}_{\lambda, x^\lambda}) : x^\lambda \in X_\lambda\}). \tag{2.12.4}$$

Since

$$ran(s_{\eta_i}^\lambda) \cap \mathcal{A}_{\lambda, x^\lambda} = \varnothing \text{ if } x^\lambda \notin dom(s_{\eta_i}^\lambda), \ \ i = 1, 2, \tag{2.12.5}$$

and

$$ran(s^{\lambda}_{\eta_i}) \cap \mathcal{A}_{\lambda,x^{\lambda}} = \{s^{\lambda}_{\eta_i}(x^{\lambda})\} \text{ if } x^{\lambda} \in dom(s^{\lambda}_{\eta_i}), \ \ i = 1,2, \tag{2.12.6}$$

the expression (2.12.4) takes the form

$$\varpi_{\lambda}(\cup\{\{s^{\lambda}_{\eta_1}(x^{\lambda})\} \times \{s^{\lambda}_{\eta_2}(x^{\lambda})\} : x^{\lambda} \in dom(s^{\lambda}_{\eta_0})\}) =$$
$$\varpi_{\lambda}(\cup\{\{(s^{\lambda}_{\eta_1}(x^{\lambda}), s^{\lambda}_{\eta_2}(x^{\lambda}))\} : x^{\lambda} \in dom(s^{\lambda}_{\eta_0})\}) =$$
$$\cup\{\{s^{\lambda}_{\eta_1}(x^{\lambda}) + s^{\lambda}_{\eta_2}(x^{\lambda})\} : x^{\lambda} \in dom(s^{\lambda}_{\eta_0})\} =$$
$$\cup\{\{(s^{\lambda}_{\eta_1} + s^{\lambda}_{\eta_2})(x^{\lambda})\} : x^{\lambda} \in dom(s^{\lambda}_{\eta_0})\}) = ran(s^{\lambda}_{\eta_0}).$$

Thus, relation (2.12.2) implies that $ran(s^{\lambda}_{\eta_0}) \subset ran(s^{\lambda}_{\eta})$.

Let $t' \in \mathcal{F}$ and $\{\eta_1, \eta_2, \eta, \eta_0\} \cup t \subset t'$. Denote by $\mathbf{H}'$ the element of $\mathrm{C}(\sim^{t'}_{\mathbf{A}})$ containing $\mathcal{A}_{\lambda}$. Therefore, $\mathbf{H}' \subset \mathbf{H}$. Then, for each $\mathcal{A}_{\mu} \in \mathbf{H}'$ we have

$$\mathcal{A}_{\mu} \sim^{t'}_{\mathbf{A}} \mathcal{A}_{\lambda}, \ \ ran(s^{\mu}_{\eta_1}) = U^{\mathcal{A}_{\mu}}_{\varepsilon_1}, \ \ ran(s^{\mu}_{\eta_2}) = U^{\mathcal{A}_{\mu}}_{\varepsilon_2}. \tag{2.12.7}$$

By condition (4) and (5) of the definitions of the families $\mathrm{R}_{\mathbf{A}}$ and $\mathrm{R}_{\mathbf{X}}$, we have

$$dom(s^{\mu}_{\eta_1}) = dom(s^{\mu}_{\eta_2}) = dom(s^{\mu}_{\eta_0}), \tag{2.12.8}$$

$$ran(s^{\mu}_{\eta_1} + s^{\mu}_{\eta_2}) = ran(s^{\mu}_{\eta_0}) \subset ran(s^{\mu}_{\eta}). \tag{2.12.9}$$

We shall prove that the sets

$$U^{\bar{\mathcal{A}}}_{\varepsilon_i}(\mathbf{H}') = \cup\{i^{\mathcal{A}_{\mu}}_{\bar{\mathcal{A}}}(ran(s^{\mu}_{\eta_i})) : \mathcal{A}_{\mu} \in \mathbf{H}'\}, \ \ i = 1,2,$$

are the required open neighbourhoods U_i of $\mathbf{a}_i$. Obviously, $\mathbf{a}_i \in U_i$. By Lemma 2.11 we have

$$\bar{\mathcal{A}} \Delta \bar{\mathcal{A}} = \cup\{\bar{\mathcal{A}}_{\mathbf{x}} \times \bar{\mathcal{A}}_{\mathbf{x}} : \mathbf{x} \in \bar{X}\} = \{i^{\mathcal{A}_{\nu}}_{\bar{\mathcal{A}}}(\mathcal{A}_{\nu,x^{\nu}}) \times i^{\mathcal{A}_{\nu}}_{\bar{\mathcal{A}}}(\mathcal{A}_{\mu,x^{\nu}}) : \mathcal{A}_{\nu} \in \mathbf{A}, x^{\nu} \in X_{\nu}\}$$

Using this relation and the relation (2.12.7) we have

$$\varpi_{\bar{\mathcal{A}}}((U^{\bar{\mathcal{A}}}_{\varepsilon_1}(\mathbf{H}') \times U^{\bar{\mathcal{A}}}_{\varepsilon_2}(\mathbf{H}')) \cap (\bar{\mathcal{A}} \boxtimes \bar{\mathcal{A}})) =$$
$$\varpi_{\bar{\mathcal{A}}}(\cup\{(i^{\mathcal{A}_{\mu}}_{\bar{\mathcal{A}}}(ran(s^{\mu}_{\eta_1})) \cap i^{\mathcal{A}_{\nu}}_{\bar{\mathcal{A}}}(\mathcal{A}_{\nu,x^{\nu}})) \times (i^{\mathcal{A}_{\xi}}_{\bar{\mathcal{A}}}(ran(s^{\xi}_{\eta_2})) \cap i^{\mathcal{A}_{\nu}}_{\bar{\mathcal{A}}}(\mathcal{A}_{\nu,x^{\nu}})) :$$
$$: \mathcal{A}_{\xi} \in \mathbf{H}', \mathcal{A}_{\nu} \in \mathbf{A}, x^{\nu} \in \bar{X}\})). \tag{2.12.10}$$

If $\mathcal{A}_{\nu} \notin \mathbf{H}'$, then the intersections in the right side of the above equality are empty. Therefore, we can suppose that $\mathcal{A}_{\nu} \in \mathbf{H}'$. In this case, relations (2.12.7)-(2.12.9) are true if we replace the letter "μ" by "ν". Let $\mathbf{a}_1 \in i^{\mathcal{A}_{\mu}}_{\bar{\mathcal{A}}}(ran(s^{\mu}_{\eta_1})) \cap i^{\mathcal{A}_{\nu}}_{\bar{\mathcal{A}}}(\mathcal{A}_{\nu,x^{\nu}})$. Then relation $\mathbf{a}_1 \in i^{\mathcal{A}_{\mu}}_{\bar{\mathcal{A}}}(ran(s^{\mu}_{\eta_1}))$ implies that there exists $a^{\mu}_1 \in ran(s^{\mu}_{\eta_1})$ and $(a^{\mu}_1, \mathcal{A}_{\mu}) \in \mathbf{a}_1$ and the relation $\mathbf{a}_1 \in i^{\mathcal{A}_{\nu}}_{\bar{\mathcal{A}}}(\mathcal{A}_{\nu,x^{\nu}})$ implies that there exists $a^{\nu}_1 \in \mathcal{A}_{\nu,x^{\nu}}$ such that $(a^{\nu}_1, \mathcal{A}_{\nu}) \in \mathbf{a}_1$ and $\pi_{\nu}(a^{\nu}_1) = x^{\nu}$. From these it follows that $\mathcal{A}_{\mu} \sim_{\mathbf{A}} \mathcal{A}_{\nu}$ and since $a^{\mu}_1 \in U^{\mathcal{A}_{\mu}}_{\varepsilon_1} = ran(s^{\mu}_{\eta_1})$ we have $a^{\nu}_1 \in U^{\mathcal{A}_{\nu}}_{\varepsilon_1} = ran(s^{\nu}_{\eta_1})$. Therefore, $a^{\nu}_1 = s^{\nu}_{\eta_1}(x^{\nu})$. Similarly, if $\mathbf{a}_2 \in i^{\mathcal{A}_{\xi}}_{\bar{\mathcal{A}}}(ran(s^{\xi}_{\eta_2})) \cap i^{\mathcal{A}_{\nu}}_{\bar{\mathcal{A}}}(\mathcal{A}_{\nu,x^{\nu}})$, then there exists $a^{\nu}_2 \in ran(s^{\nu}_{\eta_2})$ and $\pi_{\nu}(a^{\nu}_2) = x^{\nu}$. Thus, $a^{\nu}_1, a^{\nu}_2 \in \mathcal{A}_{\nu,x^{\nu}}$. Then, the equality (2.12.10) can be continued as follows:

$$\varpi_{\bar{\mathcal{A}}}(\cup\{(i^{\mathcal{A}_{\nu}}_{\bar{\mathcal{A}}}(s^{\nu}_{\eta_1}(x^{\nu})), i^{\mathcal{A}_{\nu}}_{\bar{\mathcal{A}}}(s^{\nu}_{\eta_2}(x^{\nu}))) : \mathcal{A}_{\nu} \in \mathbf{H}', x^{\nu} \in dom(s^{\nu}_{\eta_0})\}) =$$

$$\cup\{i_{\bar{\mathcal{A}}}^{\mathcal{A}_\nu}(s_{\eta_1}^\nu(x^\nu)+s_{\eta_2}^\nu(x^\nu)):\mathcal{A}_\nu\in\mathbf{H}',x^\nu\in dom(s_{\eta_0}^\nu)\}=$$
$$\cup\{i_{\bar{\mathcal{A}}}^{\mathcal{A}_\nu}((s_{\eta_1}^\nu+s_{\eta_2}^\nu)(x^\nu)):\mathcal{A}_\nu\in\mathbf{H}',x^\nu\in dom(s_{\eta_0}^\nu)\}=$$
$$\cup\{i_{\bar{\mathcal{A}}}^{\mathcal{A}_\nu}(ran(s_{\eta_0}^\nu)):\mathcal{A}_\nu\in\mathbf{H}'\}\subset$$
$$\cup\{i_{\bar{\mathcal{A}}}^{\mathcal{A}_\nu}(ran(s_{\eta}^\nu)):\mathcal{A}_\nu\in\mathbf{H}'\}=U_\varepsilon^{\bar{\mathcal{A}}}(\mathbf{H}')\subset U_\varepsilon^{\bar{\mathcal{A}}}(\mathbf{H}).$$

We note that for the first equality of the above expression we use the fact that the restriction of the mapping $i_{\bar{\mathcal{A}}}^{\mathcal{A}_\nu}$ on the set $\mathcal{A}_{\nu,x^\nu}$ is an isomorphism.

Thus, we proved relation (2.12.1), which means that the mapping $\varpi_{\bar{\mathcal{A}}}$ is continuous. Similarly, we prove that the mapping $i_{\bar{\mathcal{A}}}$ is continuous. The proof of the proposition is completed. □

Proposition 2.13. *For each point $\mathbf{a}\in\bar{\mathcal{A}}$ there exists an open neighbourhood U of $\mathbf{a}$ in $\bar{\mathcal{A}}$ such that $\bar{\pi}$ maps U homeomorphically onto an open set of $\bar{X}$.*

Proof. Let $\mathbf{a}\in\bar{\mathcal{A}}$ and $(a^\lambda,\mathcal{A}_\lambda)$ for some $\lambda\in\Lambda$. There exists an open neighbourhood of a^λ in $\mathcal{A}_\lambda$, which π_λ maps homeomorphically onto an open subset of X_λ. Since $B_0^{\mathcal{A}_\lambda}$ is a base for the open subsets of $\mathcal{A}_\lambda$, without loss of generality, we can suppose that this open neighbourhood is an element $V_\eta^{\mathcal{A}_\lambda}$ of this base. Let $t\in\mathcal{F}$ such that $\eta\in t$ and $\mathbf{H}$ be the element of $\mathrm{C}(\sim_{\mathbf{A}}^t)$ containing $\mathcal{A}_\lambda$. We prove that $U_\varepsilon^{\bar{\mathcal{A}}}(\mathbf{H})$, where $\varepsilon=\theta_{\mathbf{A}}(\theta_0(\eta))$, is the required open subset U.

By Lemma 2.8 we have
$$\bar{\pi}(U_\varepsilon^{\bar{\mathcal{A}}}(\mathbf{H}))=U_\delta^{\bar{X}}(\mathbf{L}),$$
where $\delta=\theta_{\mathbf{X}}(\eta)$ and $\mathbf{L}$ is the element of $\mathrm{C}(\mathrm{R}_{\mathbf{X}})$ corresponding to $\mathbf{H}$. First, we prove that the restriction of $\bar{\pi}$ onto the open subset $U_\varepsilon^{\bar{\mathcal{A}}}(\mathbf{H})$ of $\bar{\mathcal{A}}$ is one-to-one. Indeed, if not, there are two distinct points $\mathbf{b}_1$ and $\mathbf{b}_2$ of $U_\varepsilon^{\bar{\mathcal{A}}}(\mathbf{H})$ such that $\bar{\pi}(\mathbf{b}_1)=\bar{\pi}(\mathbf{b}_2)\equiv\mathbf{x}\in\bar{X}$, that is $\mathbf{b}_1,\mathbf{b}_2\in\bar{\mathcal{A}}_{\mathbf{x}}$. Let $(x^\nu,X_\nu)\in\mathbf{x}$ for some $\nu\in\Lambda$. By Proposition 2.11 there exist points $b_1^\nu,b_2^\nu\in\mathcal{A}_{\nu,x^\nu}$ such that $i_{\bar{\mathcal{A}}}^{\mathcal{A}_\nu}(b_1^\nu)=\mathbf{b}_1$ and $i_{\bar{\mathcal{A}}}^{\mathcal{A}_\nu}(b_2^\nu)=\mathbf{b}_2$, that is $(b_1^\nu,\mathcal{A}_\nu)\in\mathbf{b}_1$ and $(b_2^\nu,\mathcal{A}_\nu)\in\mathbf{b_2}$. We have $b_1^\nu,b_2^\nu\in U_\varepsilon^{\mathcal{A}_\nu}=V_\eta^{\mathcal{A}_\nu}=ran(s_\eta^\nu)$, $b_1^\nu\neq b_2^\nu$ and $\pi_\nu(b_1^\nu)=\pi_\nu(b_2^\nu)=x^\nu$, which contradicts the fact that the π_ν maps homeomorphically the set $ran(s_\eta^\nu)$ onto an open subset of X_ν.

Since the restriction of the mapping $\bar{\pi}$ onto the set $U_\varepsilon^{\bar{\mathcal{A}}}(\mathbf{H})$ is one-to-one we can consider the inverse mapping, denoted by $\bar{s}_{\varepsilon,\mathbf{H}}$, of the set $U_\delta^{\bar{X}}(\mathbf{L})$ onto $U_\varepsilon^{\bar{\mathcal{A}}}(\mathbf{H})$. We shall prove that $\bar{s}_{\varepsilon,\mathbf{H}}$ is continuous. Let $\mathbf{x}_1\in U_\delta^{\bar{X}}(\mathbf{L})$ and $\mathbf{a}_1=\bar{s}_{\varepsilon,\mathbf{H}}(\mathbf{x}_1)$. Let also U_1 be an arbitrary open neighbourhood of $\mathbf{a}_1$ in $U_\varepsilon^{\bar{\mathcal{A}}}(\mathbf{H})$. Since $U_\varepsilon^{\bar{\mathcal{A}}}(\mathbf{H})$ is open in $\bar{\mathcal{A}}$, without loss of generality, we can suppose that U_1 belongs to the standard base of $\bar{\mathcal{A}}$, that is it has the form $U_{\varepsilon_1}^{\bar{\mathcal{A}}}(\mathbf{H}_1)$. Moreover, we can suppose that $\varepsilon_1=\theta_{\mathbf{A}}(\theta_0(\eta_1))$ for some $\eta_1\in\tau$ and $\mathbf{H}_1\in\mathrm{C}(\sim_{\mathbf{A}}^{t_1})$, where t_1 is an element of $\mathcal{F}$ such that $t\subset t_1$. Thus, we have $\mathbf{a}_1\in U_{\varepsilon_1}^{\bar{\mathcal{A}}}(\mathbf{H}_1)\subset U_\varepsilon^{\bar{\mathcal{A}}}(\mathbf{H})$. By Lemma 2.8,
$$\bar{\pi}(U_{\varepsilon_1}^{\bar{\mathcal{A}}}(\mathbf{H}_1))=U_{\delta_1}^{\bar{X}}(\mathbf{L}_1),$$

where $\delta_1 = \theta_{\mathbf{X}}(\eta_1)$ and $\mathbf{L}_1$ is the element of $\mathrm{C}(\mathrm{R}_{\mathbf{X}})$ corresponding to $\mathbf{H}_1$. As the above, the restriction of the mapping $\bar{\pi}$ onto $U^{\bar{\mathcal{A}}}_{\varepsilon_1}(\mathbf{H}_1)$ is one-to-one and, therefore, we can consider the inverse mapping, denoted by $\bar{s}_{\varepsilon_1,\mathbf{H}_1}$, of the set $U^{\bar{X}}_{\delta_1}(\mathbf{L}_1)$ onto the set $U^{\bar{\mathcal{A}}}_{\varepsilon_1}(\mathbf{H}_1)$. The mapping $\bar{s}_{\varepsilon_1,\mathbf{H}_1}$ coincides with the restriction of the mapping $\bar{s}_{\varepsilon,\mathbf{H}}$ onto the set $U^{\bar{X}}_{\delta_1}(\mathbf{L}_1)$, that is $\bar{s}_{\varepsilon,\mathbf{H}}(U^{\bar{X}}_{\delta_1}(\mathbf{L}_1)) = U^{\bar{\mathcal{A}}}_{\varepsilon_1}(\mathbf{H}_1)$, which shows that $\bar{s}_{\varepsilon,\mathbf{H}}$ is continuous. Thus, $\bar{\pi}$ maps the set $U^{\bar{\mathcal{A}}}_{\varepsilon}(\mathbf{H})$ homeomorphically onto an open subset of $\bar{X}$. The proof of the proposition is completed. □

The final of the proof of Theorem 1.3.1 Relation (2.7.1) implies that for each $(\mathcal{A}_\lambda, \pi_\lambda, X_\lambda) \in \mathbf{S}$ the natural embedding $i^{X_\lambda}_{\bar{X}}$ of X_λ into $\bar{X}$ is the induced mapping of the natural embedding $i^{\mathcal{A}_\lambda}_{\bar{\mathcal{A}}}$ of $\mathcal{A}_\lambda$ into $\bar{\mathcal{A}}$. Proposition 2.11 shows that the embedding $i^{\mathcal{A}_\lambda}_{\bar{\mathcal{A}}}$ is proper. The rest of the proof of Theorem 1.3.1 follows immediately from Propositions 2.9, 2.11, 2.12 and 2.13.

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