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Probabilistic uniformities of uniform spaces

Jesús Rodríguez-López [a,1], *Salvador Romaguera* [a] and *Manuel Sanchis* [b]

[a] *Instituto Universitario de Matemática Pura y Aplicada, Universitat Politècnica de Valeència, 46022 Valencia, Spain* (jrlopez@mat.upv.es, sromague@mat.upv.es)

[b] *Institut Universitari de Matemàtiques i Aplicacions de Castelló (IMAC), Universitat Jaume I, Campus del Riu Sec. s/n, 12071 Castelló, Spain* (sanchis@mat.uji.es)

Abstract

Usually, fuzzy metric spaces are endowed with crisp topologies or crisp uniformities. Nevertheless, some authors have shown how to construct in this context different kinds of fuzzy uniformities like a Hutton $[0,1]$*-quasi-uniformity or a probabilistic uniformity.*

In 2010, J. Gutiérrez García, S. Romaguera and M. Sanchis [7] *proved that the category of uniform spaces is isomorphic to a category whose objects are sets endowed with a fuzzy uniform structure, i. e. a family of fuzzy pseudometrics satisfying certain conditions. We will show here that, by means of this isomorphism, we can obtain several methods to endow a uniform space with a probabilistic uniformity. Furthermore, we obtain a factorization of some functors introduced in* [6].

Keywords: *fuzzy metric space; fuzzy gauge base; probabilistic uniformity.*
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1. Introduction

The problem of finding appropriate notions for topological concepts in the fuzzy context has been a fruitful and influential area of research. In particular, the quest for finding suitable notions of fuzzy metric, fuzzy uniformity and fuzzy proximity has deserved a lot of attention during the last decades [1, 3, 5, 11, 12, 13, 14, 10, 9, 17], etc. Nevertheless, there are not too many results about how to reconcile the theory of fuzzy metric spaces with that of fuzzy uniform spaces. In crisp theory, there is a standard procedure which allows to construct a uniformity by means of a metric providing a good behaviour as from a categorical point of view as with respect to some uniform properties like precompactness and completeness. However, this procedure is not clear at all in the fuzzy theory.

In [8, 9] Höhle gave a method to construct a probabilistic uniformity and a Lowen uniformity from a probabilistic pseudometric. Recently in [6] different procedures to endow a fuzzy metric space with a probabilistic uniformity are studied. The categorical behaviour of these constructions is analyzed as well as their induced fuzzy topologies. From that study we can deduce that some of that constructions have not suitable properties since, for example, they don't preserve fuzzy uniformly continuous functions.

The present work is a continuation of the search for a standard procedure of endowing a fuzzy metric space with a probabilistic uniformity. In particular, here we are interested in the following issue. In the classical theory, there is a canonical procedure to construct a uniformity from a (pseudo)metric and this construction factorizes by means of a certain family of pseudometrics called a gauge.

$$\begin{array}{ccccc} \mathsf{Met} & & & & \mathsf{Unif} \\ (X,d) & & \longrightarrow & & (X,\mathcal{U}_d) \\ & \searrow & & \nearrow & \\ & & \mathsf{Gau} & & \\ & & (X,\mathcal{D}_d) & & \end{array}$$

We wonder whether we can obtain a similar diagram when we consider the different procedures considered in [6] of inducing a probabilistic uniformity from a fuzzy (pseudo)metric. We will show that the answer is affirmative and as a byproduct of our work we obtaing different ways of endowing a uniform space with a probabilistic uniformity.

2. Fuzzy gauge bases

Classical uniformities admit several equivalent definitions among which we can emphasize the following three: by entourages of the diagonal; by uniform covers; by pseudometrics. This last approach is based on the fact that every uniformity can be obtained as the supremum of a collection of uniformities generated by a family of pseudometrics called a *gauge* [2]. In fact, the category of uniform spaces is isomorphich to the category of gauge spaces.

In [7] it is introduced the category of fuzzy uniform spaces which can be considered as a fuzzy counterpart of the category of gauge spaces. In order to give its definition, we present other notions that will be useful later on. In the following, when we refer to a fuzzy (pseudo)metric it is in the sense of Kramosil and Michalek [13] and we presuppose that the reader is familiarized with the basic theory of fuzzy pseudometric spaces (terms and undefined concept can be consulted in [6, 7]). The category of fuzzy pseudometric spaces and uniformly continuous functions (resp. fuzzy uniformly continuous functions) will be denoted by FMet (resp. $\mathsf{FMet}_{\mathcal{F}}$).

Definition 1. A *fuzzy gauge base* on a nonempty set X is a pair $(\mathcal{B},*)$ where $*$ is a continuous t-norm and $\mathcal{B}$ is family of fuzzy pseudometrics on X with respect to the t-norm $*$ which is closed under finite infimum.

Every fuzzy gauge base $(\mathcal{B},*)$ on a nonempty set X induces a uniformity $\mathcal{U}_{\mathcal{B}}$ on X given by $\mathcal{U}_{\mathcal{B}} = \bigvee_{(M,*)\in\mathcal{B}} \mathcal{U}_M$ where $\mathcal{U}_M$ is the usual uniformity having a countable base which is associated with a fuzzy (pseudo)metric $(M,*)$ (cf. [4]). $\mathcal{U}_{\mathcal{B}}$ has as a base the family $\{U_{M,\varepsilon,t} : (M,*)\in\mathcal{B}, \varepsilon\in(0,1], t>0\}$ where $U_{M,\varepsilon,t} = \{(x,y)\in X\times X : M(x,y,t) > 1-\varepsilon\}$ (cf. [7, Proposition 3.4]). The topology generated by the uniformity $\mathcal{U}_{\mathcal{B}}$ will be denoted by $\tau(\mathcal{B})$.

Definition 2 (cf. [7])**.** Let $(X,\mathcal{B}_1,*)$ and $(Y,\mathcal{B}_2,\star)$ be two spaces endowed with two fuzzy gauge bases. A mapping $f : X \to Y$ is said to be

- *fuzzy uniformly continuous* if for every $(N,\star) \in \mathcal{B}_2$ and $t > 0$ there exist $(M,*) \in \mathcal{B}_1$ and $s > 0$ such that $M(x,y,s) \leq N(f(x),f(y),t)$ for all $x,y \in X$;
- *uniformly continuous* if for each $(N,\star) \in \mathcal{B}_2$, $\varepsilon \in (0,1]$ and $t > 0$ there exist $(M,*) \in \mathcal{B}_1$, $\delta \in (0,1]$ and $s > 0$ such that $N(f(x),f(y),t) > 1-\varepsilon$ whenever $M(x,y,s) > 1-\delta$. This is equivalent to assert that $f : (X,\mathcal{U}_{\mathcal{B}_1}) \to (Y,\mathcal{U}_{\mathcal{B}_2})$ is uniformly continuous.

Notice that every fuzzy uniformly continuous function is uniformly continuous but the converse is not true (see [15, Example 3.17]). We denote by BFGau (resp. BFGau_u) the category whose objects are the spaces endowed with a fuzzy gauge base and whose morphisms are the fuzzy uniformly continuous functions (resp. uniformly continuous functions). Of course BFGau is a subcategory of BFGau_u.

Definition 3 ([7, 15])**.** Given a fuzzy gauge base $(\mathcal{B},*)$ on a nonempty set X define:

- $\mathcal{B}^{\leq} = \{(N,*)$ fuzzy (pseudo)metric on X : there exists $(M,*) \in \mathcal{B}$ such that $M(x,y,t) \leq N(x,y,t)$ for all $x,y \in X, t > 0\}$.
- $\langle \mathcal{B} \rangle = \{(N,*)$ fuzzy (pseudo)metric on X : for all $t > 0$ there exist $(M,*) \in \mathcal{B}$ and $s > 0$ such that $M(x,y,s) \leq N(x,y,t)$ for all $x,y \in X\}$.
- $\widetilde{\mathcal{B}} = \{(N,*)$ fuzzy (pseudo)metric on X : for all $\varepsilon \in (0,1]$ and $t > 0$ there exist $s > 0, (M,*) \in \mathcal{B}$ such that $M(x,y,s) - \varepsilon \leq N(x,y,t)$ for all $x,y \in X\}$.
- $\widehat{\mathcal{B}} = \{(N,*)$ fuzzy (pseudo)metric on X : for all $\varepsilon \in (0,1]$ and $t > 0$ there exist $\delta \in (0,1], s > 0, (M,*) \in \mathcal{B}$ such that $M(x,y,s) > 1-\delta$ implies $N(x,y,t) > 1-\varepsilon\}$.

Observe that $\mathcal{B} \subseteq \mathcal{B}^{\leq} \subseteq \langle \mathcal{B} \rangle \subseteq \widetilde{\mathcal{B}} \subseteq \widehat{\mathcal{B}}$. Furthermore, if:

- $\mathcal{B}^{\leq} = \mathcal{B}$ then $(\mathcal{B},*)$ is called a *fuzzy gauge*;
- $\langle \mathcal{B} \rangle = \mathcal{B}$ then $(\mathcal{B},*)$ is called a *probabilistic uniform structure*;
- $\widetilde{\mathcal{B}} = \mathcal{B}$ then $(\mathcal{B},*)$ is called a *Lowen uniform structure*;
- $\widehat{\mathcal{B}} = \mathcal{B}$ then $(\mathcal{B},*)$ is called a *fuzzy uniform structure*.

A *fuzzy uniform space* is a triple $(X, \mathcal{M}, *)$ such that X is a nonempty set and $(\mathcal{M}, *)$ is a fuzzy uniform structure on X.

Remark 4. We notice that the mapping $\mathfrak{E}^{\leq}$: $\mathsf{BFGau} \to \mathsf{BFGau}$ leaving morphisms unchanged and such that $\mathfrak{E}^{\leq}(X, \mathcal{B}, *) = (X, \mathcal{B}^{\leq}, *)$ is an endofunctor on BFGau. This can be done for all the operators considered in the above Definition except for $\widehat{\ }$, for which we have to consider the category BFGau_u instead of BFGau.

We consider the following categories whose morphisms in all cases are the fuzzy uniformly continuous functions except in the last one where uniform continuous functions are considered:

- FGau whose objects are all spaces endowed with a fuzzy gauge;
- PSUnif whose objects are all spaces endowed with a probabilistic uniform structure;
- LSUnif whose objects are all spaces endowed with a Lowen uniform structure;
- FUnif whose objects are all fuzzy uniform spaces.

Theorem 5 ([7]). *Let $(X,\mathcal{U})$ be a uniform space and $(X, \mathcal{M}, *)$ be a fuzzy uniform space. Let us consider:*

- $(\varphi_*(\mathcal{D}_\mathcal{U}), *)$ *the fuzzy uniform structure on X given by $\varphi_*(\mathcal{D}_\mathcal{U}) = \{(M, *) : \mathcal{U}_M \subseteq \mathcal{U}\}$;*
- $\psi(\mathcal{M})$ *is the family of all pseudometrics d on X such that $\mathcal{U}_d \subseteq \mathcal{U}_\mathcal{M}$.*

Then:

(i) $\Phi_* : \mathsf{Unif} \to \mathsf{FUnif}(*)$ *is a covariant functor sending each $(X,\mathcal{U})$ to $(X, \varphi_*(\mathcal{D}_\mathcal{U}), *)$;*
(ii) $\Psi : \mathsf{FUnif}(*) \to \mathsf{Unif}$ *is a covariant functor sending each $(X, \mathcal{M}, *)$ to $(X,\mathcal{U}_\mathcal{M}) = (X,\mathcal{U}_{\psi(\mathcal{M})})$;*
(iii) $\Phi_* \circ \Psi = 1_{\mathsf{FUnif}(*)}$ *and* $\Psi \circ \Phi_* = 1_{\mathsf{Unif}}$.

3. Probabilistic uniformities

Definition 6 ([8, Definition 2.1], [11], [14])**.** A *probabilistic uniformity* on a nonempty set X is a pair $(\mathcal{U}, *)$, where $*$ is a continuous t-norm and $\mathcal{U}$ is a prefilter on $X \times X$ such that:

(PU1) $U(x,x) = 1$ for all $U \in \mathcal{U}$ and $x \in X$;
(PU2) if $U \in \mathcal{U}$ then $U^{-1} \in \mathcal{U}$ where $U^{-1}(x,y) = U(y,x)$;
(PU3) for each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V^2 \leq U$ where $V^2(x,y) = \bigvee_{z \in X} V(x,z) * V(z,y)$.

In this case, the pair $(X, \mathcal{U}, *)$ is called a *probabilistic uniform space.*

If $\mathcal{U}$ also satisfies $\bigvee_{\varepsilon \in (0,1]}(U_\varepsilon - \varepsilon) \in \mathcal{U}$ for each family $\{U_\varepsilon : \varepsilon \in (0,1]\} \subseteq \mathcal{U}$ then $(\mathcal{U}, *)$ is called a *Lowen uniformity* and $(X, \mathcal{U}, *)$ is a *Lowen uniform space.*

A function $f\colon (X, \mathcal{U}, *) \to (Y, \mathcal{V}, \star)$ between two probabilistic uniform spaces is said to be *uniformly continuous* if $(f \times f)^{-1}(V) \in \mathcal{U}$ for all $V \in \mathcal{V}$, i.e. for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that

$$U(x,y) \leq V(f(x), f(y)) \text{ for all } x, y \in X.$$

We denote by PUnif (resp. LUnif) the category of probabilistic uniform spaces (resp. Lowen uniform spaces) and uniformly continuous functions. For a fixed continuous t-norm, $\mathsf{PUnif}(*)$ (resp. $\mathsf{LUnif}(*)$) is the full subcategory of PUnif (resp. LUnif) whose objects are the probabilistic uniform spaces (resp. Lowen uniform spaces) with respect to $*$.

Theorem 7 ([14])**.** *Let X be a nonempty set, $\mathcal{U}$ be a uniformity on X and $(\mathcal{U}, *)$ be a Lowen uniformity on X. Define*

$$\omega(\mathcal{U}) = \{U \in I^{X \times X} : U^{-1}((\alpha, 1]) \in \mathcal{U} \text{ for all } \alpha \in I_1\} \quad \text{and}$$
$$\iota(\mathcal{U}) = \{U^{-1}((\alpha, 1]) : U \in \mathcal{U},\ \alpha \in I_1\}.$$

Then the functor $\omega_*\colon \mathsf{Unif} \to \mathsf{LUnif}(*)$ given by $\omega_*((X, \mathcal{U})) = (X, \omega(\mathcal{U}), *)$ and which leaves morphisms unchanged is fully faithful while the functor $\iota\colon \mathsf{LUnif} \to \mathsf{Unif}$ given by $\iota((X, \mathcal{U}, *)) = (X, \iota(\mathcal{U}))$ and which leaves morphisms unchanged is *faithful. Furthermore,* $\iota \circ \omega_* = 1_{\mathsf{Unif}}$ *so* Unif *is isomorphic to a full subcategory of* LUnif$(*)$.

Remark 8. It is proved in [16] that LUnif is a coreflective full subcategory of PUnif and the coreflector is the functor $\mathcal{S} : \mathsf{PUnif} \to \mathsf{LUnif}$ which leaves morphisms unchanged and which assigns to every probabilistic uniformity $(\mathcal{U}, *)$ its saturation $(\widetilde{\mathcal{U}}, *)$ where $\widetilde{\mathcal{U}} = \{\bigvee_{\varepsilon\in(0,1]}(U_\varepsilon - \varepsilon) : (U_\varepsilon)_{\varepsilon\in(0,1]} \in \mathcal{U}^{(0,1]}\}$.

4. Probabilistic uniformities on a uniform space

Next we propose some methods to endow a uniform space (or equivalently a fuzzy uniform space) with a probabilistic uniformity.

Proposition 9. *Consider the mappings*

$$\Lambda_s, \Upsilon_s : \mathsf{BFGau} \to \mathsf{PUnif}, \qquad \Gamma_s, \omega_s : \mathsf{BFGau}_u \to \mathsf{PUnif}$$

leaving morphisms unchanged and acting on objects as:

(1) $\Lambda_s(X, \mathcal{B}, *) = (X, \mathcal{U}_\mathcal{B}, *)$ *where* $(\mathcal{U}_\mathcal{B}, *)$ *is the probabilistic uniformity which has as a base the family* $\{U^M_{\varepsilon,t} : \varepsilon \in (0,1],\ t > 0, (M,*) \in \mathcal{B}\}$ *where*

$$U^M_{\varepsilon,t}(x,y) = (1-\varepsilon) \to M(x,y,t) = \bigvee\{\lambda \in [0,1] : (1-\varepsilon) * \lambda \leq M(x,y,t)\}$$

for all $x, y \in X$.

(2) $\Upsilon_s(X, \mathcal{B}, *) = (X, \mathcal{U}^H_\mathcal{B}, *)$ *where* $(\mathcal{U}^H_\mathcal{B}, *)$ *is the probabilistic uniformity which has as a base the family* $\{M_t : t > 0, (M,*) \in \mathcal{B}\}$ *and* $M_t(x,y) = M(x,y,t)$ *for all* $x, y \in X$.

(3) $\Gamma_s(X, \mathcal{B}, *) = (X, \mathcal{U}^{01}_\mathcal{B}, *)$ *where* $(\mathcal{U}^{01}_\mathcal{B}, *)$ *is the probabilistic uniformity which has as a base the family* $\{1_U : U \in \mathcal{U}_\mathcal{B}\}$ *and* 1_U *is the characteristic function of* U.

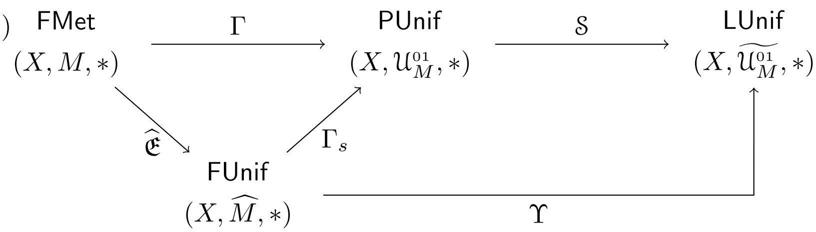
(4) $\omega_s(X, \mathcal{B}, *) = (X, \omega(\mathcal{U}_\mathcal{B}), *)$.

Then $\Gamma_s, \omega_s, \Lambda_s, \Lambda^H_s$ are covariant functors.

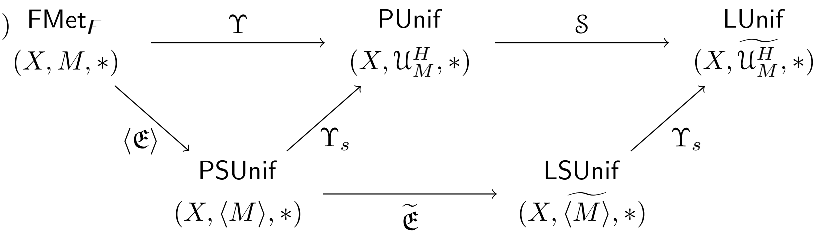
Remark 10. Notice that composing the above mappings with the functor Φ_* (see Theorem 5) we obtain several methods to construct a probabilistic uniformity from a crisp uniformity.

In [6] several functors from FMet to PUnif were considered. It is natural to wonder if they factorizes by means of some subcategory of BFGau_u.

Proposition 11. *The following diagrams commute:*

(1)


where Γ *is the restriction of* Γ_s *to the full subcategory* FMet *of* BFGau_u.

(2)


where Υ *is the restriction of* Υ_s *to the full subcategory* FMet_F *of* BFGau.

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