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# $e_c$-Filters and $e_c$-ideals in the functionally countable subalgebra of $C^*(X)$

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Abstract

*The purpose of this article is to study and investigate $e_c$-filters on $X$ and $e_c$-ideals in $C_c^*(X)$ in which they are in fact the counterparts of $z_c$-filters on $X$ and $z_c$-ideals in $C_c(X)$ respectively. We show that the maximal ideals of $C_c^*(X)$ are in one-to-one correspondence with the $e_c$-ultrafilters on $X$. In addition, the sets of $e_c$-ultrafilters and $z_c$-ultrafilters are in one-to-one correspondence. It is also shown that the sets of maximal ideals of $C_c(X)$ and $C_c^*(X)$ have the same cardinality. As another application of the new concepts, we characterized maximal ideals of $C_c^*(X)$. Finally, we show that whether the space $X$ is compact, a proper ideal $I$ of $C_c(X)$ is an $e_c$-ideal if and only if it is a closed ideal in $C_c(X)$ if and only if it is an intersection of maximal ideals of $C_c(X)$.*

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## 1. Introduction

All topological spaces are completely regular Hausdorff spaces and we shall assume that the reader is familiar with the terminology and basic results of [6]. Given a topological space $X$, we let $C(X)$ denote the ring of all real-valued continuous functions defined on $X$. $C_c(X)$ is the subalgebra of $C(X)$ consisting of functions with countable image and $C_c^*(X)$ is its subalgebra consisting of bounded functions. In fact, $C_c^*(X) = C_c(X) \cap C^*(X)$, where elements of $C^*(X)$

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are bounded functions of $C(X)$. Recall that for $f \in C(X), Z(f)$ denotes its zero-set:

$$Z(f) = \{x \in X : f(x) = 0\}.$$

The set-theoretic complement of a zero-set is known as a cozero-set and we denote this set by $\operatorname{coz}(f)$. Let us put $Z_c(X) = \{Z(f) : f \in C_c(X)\}$ and $Z_c^*(X) = \{Z(g) : g \in C_c^*(X)\}$. These two latter sets are in fact equal, since $Z(f) = Z(\frac{f}{1+|f|})$, where $f \in C_c(X)$. A nonempty subfamily $\mathcal{F}$ of $Z_c(X)$ is called a $z_c$-filter if it is a filter on $X$. If $I$ is an ideal in $C_c(X)$ and $\mathcal{F}$ is a $z_c$-filter on $X$ then, we denote $Z_c[I] = \{Z(f) : f \in I\}$, $\cap Z_c[I] = \cap\{Z(f) : f \in I\}$ and $Z_c^{-1}[\mathcal{F}] = \{f : Z(f) \in \mathcal{F}\}$. We see that $Z_c[I]$ is a $z_c$-filter and $Z_c^{-1}[Z_c[I]] \supseteq I$. If the equality holds, then $I$ is called a $z_c$-ideal. Moreover, $Z_c^{-1}[\mathcal{F}]$ is a $z_c$-ideal and we always have $Z_c[Z_c^{-1}[\mathcal{F}]] = \mathcal{F}$. So maximal ideals in $C_c(X)$ are $z_c$-ideals. In [5], a Huasdorff space $X$ is called countably completely regular (briefly, $c$-completely regular) if whenever $F$ is a closed subset of $X$ and $x \notin F$, there exists $f \in C_c(X)$ such that $f(x) = 0$ and $f(F) = 1$. In addition, two closed sets $A$ and $B$ of $X$ are also called countably separated (in brief, $c$-separated) if there exists $f \in C_c(X)$ with $f(A) = 0$ and $f(B) = 1$. $c$-completely regular and zero-dimensional spaces are the same, see Theorem 1.1.

If we let $M_p^c = \{f \in C_c(X) : f(p) = 0\}$ $(p \in X)$, then the ring isomorphism $\frac{C_c(X)}{M_p^c} \cong \mathbb{R}$ gives that $M_p^c$ is a maximal ideal, in fact, $M_p^c$ is a fixed maximal ideal. Moreover, $\cap Z_c[M_p^c] = \{p\}$.

Our concentration is on the zero-dimensional spaces since in [5] the authors proved that for any space $X$ there is a zero-dimensional Hausdorff space $Y$ such that $C_c(X)$ and $C_c(Y)$ are isomorphic as rings, see Theorem 1.2.

In section 2, we study and investigate the $e_c$-filters on $X$ and $e_c$-ideals in $C_c^*(X)$ which they are in fact the counterpart of [6, 2L]. We show that the maximal ideals of $C_c^*(X)$ are in one-to-one correspondence with the $e_c$-ultrafilters on $X$. Moreover, the sets of $e_c$-ultrafilters and $z_c$-ultrafilters are in one-to-one correspondence. By using the latter facts, it is shown that the sets of maximal ideals of $C_c(X)$ and $C_c^*(X)$ have the same cardinality. Finally, maximal ideals of $C_c^*(X)$ are characterized based on these concepts. In Section 3, our concentration is on the uniform norm topology on $C_c^*(X)$ which is the restriction of the uniform norm topology on $C^*(X)$. It is shown that whenever the space $X$ is compact, a proper ideal $I$ of $C_c(X)$ is an $e_c$-ideal if and only if it is a closed ideal in $C_c(X)$ if and only if it is an intersection of maximal ideals of $C_c(X)$.

We recite the following results from [5].

**Theorem 1.1** ([5, Proposition 4.4]). *Let $X$ be a topological space. Then, $X$ is a zero-dimensional space (i.e., a $T_1$-space with a base consisting of clopen sets) if and only if $X$ is $c$-completely regular space.*

**Theorem 1.2** ([5, Theorem 4.6]). *Let $X$ be any topological space (not necessarily completely regular). Then, there is a zero-dimensional space $Y$ which is a continuous image of $X$ with $C_c(X) \cong C_c(Y)$ and $C^F(X) \cong C^F(Y)$.*

*Remark* 1.3 ([5, Remark 7.5]). There is a topological space $X$, such that there is no space $Y$ with $C_c(X) \cong C(Y)$.

The following results are the known facts about $C_c(X)$ and we are seeking to get similar results for $C_c^*(X)$.

**Proposition 1.4.** *Let $I$ be a proper ideal in $C_c(X)$ and $\mathcal{F}$ a $z_c$-filter on $X$. Then:*

(i) *$Z_c[I]$ is a $z_c$-filter and $Z_c^{-1}[\mathcal{F}]$ is a $z_c$-ideal of $C_c(X)$.*
(ii) *If $I$ is maximal then $Z_c[I]$ is a $z_c$-ultrafilter, and the converse holds if $I$ is a $z_c$-ideal.*
(iii) *$\mathcal{F}$ is a $z_c$-ultrafilter if and only if $Z_c^{-1}[\mathcal{F}]$ is a maximal ideal.*
(iv) *If $\mathcal{F}$ is a $z_c$-ultrafilter and $Z \in Z_c(X)$ meets each element of $\mathcal{F}$, then $Z \in \mathcal{F}$.*

**Corollary 1.5.** *There is a one-to-one correspondence $\psi$ between the sets of $z_c$-ideals of $C_c(X)$ and $z_c$-filters on $X$, defined by $\psi(I) = Z_c[I]$. In particular, the restriction of $\psi$ to the set of maximal ideals is a one-to-one correspondence between the sets of maximal ideals of $C_c(X)$ and $z_c$-ultrafilters on $X$.*

## 2. $e_c$-filters on $X$ and $e_c$-ideals in $C_c^*(X)$

For $f \in C_c^*(X)$ and $\epsilon > 0$, we define

$$E_\epsilon^c(f) = f^{-1}([-\epsilon, \epsilon]) = \{x \in X : |f(x)| \leq \epsilon\}.$$

Each such set is a zero set, since it is equal to $Z((|f| - \epsilon) \vee 0)$. Conversely, every zero set is also of this form, since for $g \in C_c^*(X)$ we have $Z(g) = E_\epsilon^c(|g| + \epsilon)$. For a nonempty subset $I$ of $C_c^*(X)$ we denote $E_\epsilon^c[I] = \{E_\epsilon^c(f) : f \in I\}$, and $E_c(I) = \bigcup_\epsilon E_\epsilon^c[I]$. Moreover, if $\mathcal{F}$ is a nonempty subfamily of $Z_c^*(X)$, then we define $E_\epsilon^{c-1}[\mathcal{F}] = \{f \in C_c^*(X) : E_\epsilon^c(f) \in \mathcal{F}\}$ and $E_c^{-1}(\mathcal{F}) = \bigcap_\epsilon E_\epsilon^{c-1}[\mathcal{F}]$. So we have $E_c(I) = \{E_\epsilon^c(f) : f \in I \text{ and } \epsilon > 0\}$, and $E_c^{-1}(\mathcal{F}) = \{f \in C_c^*(X) : E_\epsilon^c(f) \in \mathcal{F}, \text{ for all } \epsilon\}$. Moreover, $E_c^{-1}(E_c(I)) = \{g \in C_c^*(X) : E_\delta^c(g) \in E_c(I), \text{for all } \delta > 0\}$ and $E_c(E_c^{-1}(\mathcal{F})) = \{E_\epsilon^c(f) : E_\delta^c(f) \in \mathcal{F}, \text{for all } \delta > 0\}$.

The next result is now immediate.

**Corollary 2.1.** *The following statements hold.*

(i) *$I \subseteq E_c^{-1}(E_c(I))$ and $E_c(E_c^{-1}(\mathcal{F})) \subseteq \mathcal{F}$.*
(ii) *The mappings $E_c$ and $E_c^{-1}$ preserve the inclusion.*
(iii) *If $f \in I$ then for each positive integer $n$, $E_\epsilon^c(f) = E_{\epsilon^n}^c(f^n)$.*
(iv) *If $I$ is an ideal, then $E_c(I)$ is a $z_c$-filter.*

*Proof.* The proofs of (i), (ii) and (iii) are clear. (iv). This is presented in the proof of Proposition 2.5. □

Examples 2.2 and 2.3 below show that the inclusions in (i) of the above corollary may be strict even when $I$ is an ideal and $\mathcal{F}$ is a $z_c$-filter.

**Example 2.2.** Let $X$ be the discrete space $\mathbb{N}\times\mathbb{N}$, $f(m,n)=\frac{1}{mn}$ and $I$ the ideal in $C_c^*(X)(=C^*(X))$ generated by $f^2$. Obviously $f\notin I$. Since $\{x\in X : f(x)\leq \epsilon\}=\{x\in X : f^2(x)\leq\epsilon^2\}$, we have $E_\epsilon^c(f)\in E_c(I)$. So $I\subsetneqq E_c^{-1}(E_c(I))$.

**Example 2.3.** Let $X$ be the zero-dimensional space $\mathbb{Q}\times\mathbb{Q}$, where $\mathbb{Q}$ is the set of rational numbers, and $\mathcal{F}=\{Z\in Z_c(X) : (0,0)\in Z\}$. Then $\mathcal{F}$ is a $z_c$-filter on $X$. Now, if we define $f(x,y)=\frac{|x|+|y|}{1+|x|+|y|}$ then $f\in C_c^*(X)(=C^*(X))$ and $Z(f)=\{(0,0)\}$. Given $\epsilon>0$ and $g=f+\epsilon$, so we have $E_\epsilon^c(g)=\{(0,0)\}$. If we take $0<\delta<\epsilon$ then $E_\delta^c(g)=\varnothing$. Hence $E_\epsilon^c(g)$ is not contained in $E_c(E_c^{-1}(\mathcal{F}))$. Therefore the latter set is contained in $\mathcal{F}$ properly, which gives the result.

**Definition 2.4.** A $z_c$-filter $\mathcal{F}$ is called an $e_c$-filter if $\mathcal{F}=E_c(E_c^{-1}(\mathcal{F}))$, or equivalently, whenever $Z\in\mathcal{F}$ then there exist $f\in C_c^*(X)$ and $\epsilon>0$ such that $Z=E_\epsilon^c(f)$ and $E_\delta^c(f)\in\mathcal{F}$, for each $\delta>0$.

**Proposition 2.5.** *If $I$ is a proper ideal in $C_c^*(X)$, then $E_c(I)$ is an $e_c$-filter.*

*Proof.* First, we show that $E_c(I)$ is a $z_c$-filter, i.e., it satisfies the following conditions.

(i) $\varnothing\notin E_c(I)$.

(ii) $E_\epsilon^c(f), E_\delta^c(g)\in E_c(I)$, then $E_\epsilon^c(f)\cap E_\delta^c(g)\in E_c(I)$.

(iii) $E_\epsilon^c(f)\in E_c(I)$, $Z\in Z_c(X)$ with $Z\supseteq E_\epsilon^c(f)$, then $Z\in E_c(I)$.

(i). Suppose that for some $\epsilon>0$ and $f\in I$, $E_\epsilon^c(f)=\varnothing$. So $\epsilon<|f|$, which yields $f$ is a bounded away from zero. Hence $I$ contains the unit $f$, which is impossible. (ii). This is equivalent to say that if $E_\epsilon^c(f), E_\delta^c(g)\in E_c(I)$, then $E_\epsilon^c(f)\cap E_\delta^c(g)$ contains a member of $E_c(I)$. Suppose that $f', g'\in I$ and $\epsilon', \delta'>0$ such that $E_\epsilon^c(f)=E_{\epsilon'}^c(f')$ and $E_\delta^c(g)=E_{\delta'}^c(g')$. Without loss of generality, we may suppose that $\delta'<\epsilon'<1$. Hence $f'^2+g'^2\in I$ and $E_{\delta'^2}^c(f'^2+g'^2)\subseteq E_\epsilon^c(f)\cap E_\delta^c(g)$, which gives the result. (iii). Assume that $E_\epsilon^c(f)\subseteq Z(f')$, where $f\in I$ and $f'\in C_c^*(X)$. Since $E_\epsilon^c(f)=E_{\epsilon^2}^c(f^2)$ and $Z(f')=Z(|f'|)$, we can suppose that $f\geq 0$ and $f'\geq 0$. Now, define

$$g(x)=\begin{cases}1, & \text{if } x\in E_\epsilon^c(f)\\ f'(x)+\frac{\epsilon}{f(x)}, & \text{if } x\in X\setminus \text{int}E_\epsilon^c(f).\end{cases}$$

So $g$ is continuous, since it is continuous on two closed sets whose union is $X$, in fact, $g\in C_c^*(X)$. Note that $fg\in I$ and

$$(fg)(x)=\begin{cases}f(x), & \text{if } x\in E_\epsilon^c(f)\\ (ff')(x)+\epsilon, & \text{if } x\in X\setminus \text{int}E_\epsilon^c(f).\end{cases}$$

It is easily seen that $Z(f')=E_\epsilon^c(fg)$. So $Z(f')\in E_c(I)$. This shows that $E_c(I)$ is a $z_c$-filter. Now, apply (i) and (ii) of Corollary 2.1 for the ideal $I$ and the $z_c$-filter $E_c(I)$, to get the inclusions $E_c(I)\subseteq E_c(E_c^{-1}(E_c(I)))$ and $E_c(E_c^{-1}(E_c(I)))\subseteq E_c(I)$, which yields $E_c(I)$ is an $e_c$-filter. □

**Definition 2.6.** An ideal $I$ in $C_c^*(X)$ is called $e_c$-ideal if $I=E_c^{-1}(E_c(I))$, or equivalently, if $f\in C_c^*(X)$ and $E_\epsilon^c(f)\in E_c(I)$ for all $\epsilon$, then $f\in I$.

**Proposition 2.7.** *If $\mathcal{F}$ is a $z_c$-filter, then $E_c^{-1}(\mathcal{F})$ is an $e_c$-ideal in $C_c^*(X)$.*

*Proof.* Let $f, g \in E_c^{-1}(\mathcal{F})$, $h \in C_c^*(X)$ and let $u$ be an upper bound for $h$ and $\epsilon > 0$. Then $E^c_{\frac{\epsilon}{2}}(f)$, $E^c_{\frac{\epsilon}{2}}(g)$ and hence $E^c_{\frac{\epsilon}{2}}(f) \cap E^c_{\frac{\epsilon}{2}}(g)$ belong to $\mathcal{F}$. Hence $E^c_{\frac{\epsilon}{2}}(f) \cap E^c_{\frac{\epsilon}{2}}(g) \subseteq E^c_\epsilon(f+g)$ implies that $E^c_\epsilon(f+g) \in \mathcal{F}$, or equivalently, $f+g \in E_c^{-1}(\mathcal{F})$. Moreover, $E^c_{\frac{\epsilon}{u}}(f) \subseteq E^c_\epsilon(fh)$ implies $fh \in E_c^{-1}(\mathcal{F})$. Therefore $E_c^{-1}(\mathcal{F})$ is ideal. In view of Corollary 2.1, we have $E_c^{-1}(\mathcal{F}) \subseteq E_c^{-1}(E_c(E_c^{-1}(\mathcal{F}))) \subseteq E_c^{-1}(\mathcal{F})$ and so the equality holds, i.e., $E_c^{-1}(\mathcal{F})$ is an $e_c$-ideal. □

**Corollary 2.8.** *Maximal ideals of $C_c^*(X)$ and an arbitrary intersection of them are $e_c$-ideals.*

*Proof.* Let $M$ be a maximal ideal of $C_c^*(X)$. If $E_c^{-1}(E_c(M))$ is not a proper $e_c$-ideal, then it contains the constant function 1 and $E^c_\epsilon(1) = \varnothing \in E_c(M)$ $(0 < \epsilon < 1)$ which is impossible, see Propositions 2.5 and 2.7. Hence $M = E_c^{-1}(E_c(M))$, i.e., $M$ is an $e_c$-ideal. The second part is obtained by this fact, the fact that the intersection of a family of maximal ideals is an ideal contained in each of them and (ii) of Corollary 2.1. □

The next corollary is an immediate result of Propositions 2.5 and 2.7.

**Corollary 2.9.** *The correspondence $I \mapsto E_c(I)$ is one-one from the set of $e_c$-ideals in $C_c^*(X)$ onto the set of $e_c$-filters on $X$.*

**Lemma 2.10.**
(i) *Let $I$ and $J$ be ideals in $C_c^*(X)$ and $J$ an $e_c$-ideal. Then $I \subseteq J$ if and only if $E_c(I) \subseteq E_c(J)$.*
(ii) *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be $z_c$-filters on $X$ and $\mathcal{F}_1$ an $e_c$-filter. Then $\mathcal{F}_1 \subseteq \mathcal{F}_2$ if and only if $E_c^{-1}(\mathcal{F}_1) \subseteq E_c^{-1}(\mathcal{F}_2)$.*

*Proof.* It is straightforward. □

**Proposition 2.11.** *Let $I$ be an ideal in $C_c^*(X)$ and $\mathcal{F}$ a $z_c$-filter on $X$. Then:*
(i) *$E_c^{-1}(E_c(I))$ is the smallest $e_c$-ideal containing $I$.*
(ii) *$E_c(E_c^{-1}(\mathcal{F}))$ is the largest $e_c$-filter contained in $\mathcal{F}$.*

*Proof.* (i). Propositions 2.5 and 2.7 respectively show that $E_c(I)$ is an $e_c$-filter and $E_c^{-1}(E_c(I))$ is an $e_c$-ideal. Now, suppose that $\mathcal{K}$ is an $e_c$-ideal containing $I$. So $E_c^{-1}(E_c(I)) \subseteq E_c^{-1}(E_c(\mathcal{K})) = \mathcal{K}$. Hence we are done. (ii). This is proved similarly. □

The next theorem plays an important role in many of the following results.

**Theorem 2.12.** *Let $\mathcal{A}$ be a $z_c$-ultrafilter. Then a zero set $Z$ meets every element of $E_c(E_c^{-1}(\mathcal{A}))$ if and only if $Z \in \mathcal{A}$.*

*Proof.* Since $\mathcal{A}$ is a filter and $E_c(E_c^{-1}(\mathcal{A})) \subseteq \mathcal{A}$, the sufficient condition is evident. For the necessary condition, it is recalled at first that if $Z$ meets every element of $\mathcal{A}$ then $Z \in \mathcal{A}$, see (iv) of Proposition 1.4. Now, we claim that if $Z$ meets every element of $E_c(E_c^{-1}(\mathcal{A}))$ as a particular subfamily of $\mathcal{A}$, then also $Z \in \mathcal{A}$. Otherwise, for some $Z' \in \mathcal{A}$, $Z \cap Z' = \varnothing$. Since the closed sets $Z$ and

$Z'$ are $c$-completely separated, there is $f \in C_c^*(X)$ (in fact $0 \le f \le 1$) such that $f(Z) = 1$ and $f(Z') = 0$. Notice that $Z' \subseteq Z(f) \subseteq E_\epsilon^c(f)$, for all $\epsilon$, and; $E_\epsilon^c(f) \in \mathcal{A}$, since $Z' \in \mathcal{A}$. So $E_\epsilon^c(f) \in E_c(E_c^{-1}(\mathcal{A}))$. Now, if $\epsilon$ is taken less than 1, then $Z \cap E_\epsilon^c(f) = \varnothing$ which contradicts with our assumption of $Z$. So $Z \in \mathcal{A}$ and the proof is complete. □

The following proposition shows that, as $Z_c^{-1}(\mathcal{A})$ is a maximal ideal in $C_c(X)$, $E_c^{-1}(\mathcal{A})$ is also a maximal ideal in $C_c^*(X)$, where $\mathcal{A}$ is a $z_c$-ultrafilter on $X$.

**Proposition 2.13.** *Let $\mathcal{A}$ be a $z_c$-ultrafilter on $X$. Then:*

(i) *$E_c^{-1}(\mathcal{A})$ is a maximal ideal.*
(ii) *$E_c^{-1}(\mathcal{A})$ is an $e_c$-ideal.*
(iii) *$E_c^{-1}(\mathcal{A}) = E_c^{-1}(E_c(E_c^{-1}(\mathcal{A})))$.*

*Proof.* (i). Let $M$ be a maximal ideal of $C_c^*(X)$ containing $E_c^{-1}(\mathcal{A})$. Hence $E_c(E_c^{-1}(\mathcal{A})) \subseteq E_c(M)$. Since every element of $E_c(M)$ meets every element of $E_c(E_c^{-1}(\mathcal{A}))$, Theorem 2.12 gives $E_c(M) \subseteq \mathcal{A}$. So $M = E_c^{-1}(E_c(M)) \subseteq E_c^{-1}(\mathcal{A})$ and hence $M = E_c^{-1}(\mathcal{A})$. (ii). It follows by (i). (iii). Since the maximal ideal $E_c^{-1}(\mathcal{A})$ is contained in the proper ideal $E_c^{-1}(E_c(E_c^{-1}(\mathcal{A})))$, the result now holds. □

An $e_c$-ultrafilter on $X$ is meant a maximal $e_c$-filter, i.e., one not contained in any other $e_c$-filter. As usual, every $e_c$-filter $\mathcal{F}$ is contained in an $e_c$-ultrafilter. This is obtained by considering the collection of all $e_c$-filters containing $\mathcal{F}$ and the use of the Zorn's lemma, where the partially ordered relation on $\mathcal{F}$ is inclusion.

**Proposition 2.14.** *Let $M$ be an ideal in $C_c^*(X)$ and $\mathcal{F}$ a $z_c$-filter on $X$. Then:*

(i) *If $M$ is a maximal ideal then $E_c(M)$ is an $e_c$-ultrafilter.*
(ii) *If $\mathcal{F}$ is an $e_c$-ultrafilter then $E_c^{-1}(\mathcal{F})$ is a maximal ideal.*
(iii) *If $M$ is an $e_c$-ideal, then $M$ is maximal if and only if $E_c(M)$ is an $e_c$-ultrafilter.*
(iv) *If $\mathcal{F}$ is an $e_c$-filter, then $\mathcal{F}$ is $e_c$-ultrafilter if and only if $E_c^{-1}(\mathcal{F})$ is a maximal ideal.*

*Proof.* (i). Note that $M = E_c^{-1}(E_c(M))$. Let $\mathcal{F}'$ be an $e_c$-ultrafilter containing $E_c(M)$, then $M \subseteq E_c^{-1}(\mathcal{F}')$ and hence $M = E_c^{-1}(\mathcal{F}')$. Therefore $E_c(M) = E_c(E_c^{-1}(\mathcal{F}')) = \mathcal{F}'$, which yields the result. (ii). Let $M$ be a maximal ideal of $C_c^*(X)$ containing $E_c^{-1}(\mathcal{F})$. Then $\mathcal{F} \subseteq E_c(M)$. Hence $\mathcal{F} = E_c(M)$ and so $E_c^{-1}(\mathcal{F}) = M$. The proofs of (iii) and (iv) are similarly done and further details are omitted. □

**Corollary 2.15.** *There is a one-to-one correspondence $\psi$ between the sets of maximal ideals of $C_c^*(X)$ and $e_c$-ultrafilters on $X$, defined by $\psi(M) = E_c(M)$.*

**Proposition 2.16.** *Let $\mathcal{A}$ be a $z_c$-ultrafilter. Then it is the unique $z_c$-ultrafilter containing $E_c(E_c^{-1}(\mathcal{A}))$, and also $E_c(E_c^{-1}(\mathcal{A}))$ is the unique $e_c$-ultrafilter contained in $\mathcal{A}$. Hence every $e_c$-ultrafilter is contained in unique $z_c$-ultrafilter.*

*Proof.* Let $\mathcal{B}$ be a $z_c$-ultrafilter containing $E_c(E_c^{-1}(\mathcal{A}))$ and $Z \in \mathcal{B}$. Since $Z$ meets every element of $E_c(E_c^{-1}(\mathcal{A}))$, Theorem 2.12 gives $\mathcal{B} \subseteq \mathcal{A}$ and hence $\mathcal{B} = \mathcal{A}$. So the first part of the proposition holds. Now, let $\mathcal{K}$ be an $e_c$-ultrafilter contained in $\mathcal{A}$. Then $\mathcal{K} = E_c(E_c^{-1}(\mathcal{K})) \subseteq E_c(E_c^{-1}(\mathcal{A}))$. Since the latter set is an $e_c$-filter, the inclusion cannot be proper, i.e., $\mathcal{K} = E_c(E_c^{-1}(\mathcal{A}))$. Hence the result is obtained. □

**Corollary 2.17.** *The $z_c$-ultrafilters are in one-to-one correspondence with the $e_c$-ultrafilters.*

*Proof.* Consider the mapping $\psi$ from the set of $z_c$-ultrafilters into the set of $e_c$-ultrafilters defined by $\psi(\mathcal{A}) = E_c(E_c^{-1}(\mathcal{A}))$. If $\psi(\mathcal{A}) = \psi(\mathcal{B})$, then we have that $E_c(E_c^{-1}(\mathcal{A})) = E_c(E_c^{-1}(\mathcal{B}))$ and it is contained in both $\mathcal{A}$ and $\mathcal{B}$. So each element of $\mathcal{B}$ meets each element of $E_c(E_c^{-1}(\mathcal{A}))$. Now, Theorem 2.12 gives $\mathcal{B} \subseteq \mathcal{A}$. Similarly, $\mathcal{A} \subseteq \mathcal{B}$. Therefore $\psi$ is one-one. Let $\mathcal{K}$ be an $e_c$-ultrafilter and $\mathcal{A}$ the unique $z_c$-ultrafilter containing it (Proposition 2.16). Then $\mathcal{K} = E_c(E_c^{-1}(\mathcal{A}))$ and hence $\psi(\mathcal{A}) = \mathcal{K}$. Therefore $\psi$ is onto. □

Our next two theorems are applications that are based on the concepts of $e_c$-filters and $e_c$-ideals. In the first result (Theorem 2.18) we show that the maximal ideals of $C_c(X)$ are in one-to-one correspondence with those ones of $C_c^*(X)$ and the second result (Theorem 2.20) involves characterization of maximal ideals of $C_c^*(X)$.

**Theorem 2.18.** *Let $\mathcal{M}$ (resp. $\mathcal{M}^*$) be the set of maximal ideals of $C_c(X)$ (resp. $C_c^*(X)$). Then $\mathcal{M}$ and $\mathcal{M}^*$ have the same cardinality.*

*Proof.* If $M \in \mathcal{M}$ then $Z_c[M]$ is a $z_c$-ultrafilter and hence $E_c^{-1}(Z_c[M]) \in \mathcal{M}^*$, see Propositions 1.4 and 2.13. Define

$$\varphi : \mathcal{M} \to \mathcal{M}^* \text{ which } M \mapsto E_c^{-1}(Z_c[M]).$$

If $\varphi(M) = \varphi(M')$ then $E_c(E_c^{-1}(Z_c[M])) = E_c(E_c^{-1}(Z_c[M']))$ and it is contained in both $Z_c[M]$ and $Z_c[M']$. Since each element of $Z_c[M']$ meets each element of $E_c(E_c^{-1}(Z_c[M]))$, Theorem 2.12 yields $Z_c[M'] \subseteq Z_c[M]$. Similarly, $Z_c[M] \subseteq Z_c[M']$. Therefore $M = M'$. This verifies $\varphi$ is one-one. To show that $\varphi$ is onto, suppose that $M^* \in \mathcal{M}^*$. Hence $E_c(M^*)$ is an $e_c$-ultrafilter (Proposition 2.14). Now, let $\mathcal{A}$ be the unique $z_c$-ultrafilter containing $E_c(M^*)$ (Proposition 2.16), then $Z_c^{-1}[\mathcal{A}]$ is a maximal ideal in $C_c(X)$ (Proposition 1.4) and $M^* = E_c^{-1}(\mathcal{A})$. Recall that if $\mathcal{F}$ is a $z_c$-filter, then we always have $Z_c[Z_c^{-1}[\mathcal{F}]] = \mathcal{F}$ and $E_c(E_c^{-1}(\mathcal{F})) \subseteq \mathcal{F}$, but the equality occurs if $\mathcal{F}$ is an $e_c$-filter. Now, if we let $M = Z_c^{-1}[\mathcal{A}]$ then $\varphi(M) = E_c^{-1}(Z_c[M]) = E_c^{-1}(\mathcal{A}) = M^*$. Hence $\varphi$ is onto, which it completes the proof. □

*Remark* 2.19. Combining Corollaries 1.5, 2.15 and 2.17 gives another proof of the above theorem.

**Theorem 2.20.** *Let $M$ be an ideal in $C_c^*(X)$. Then $M$ is maximal if and only if whenever $f \in C_c^*(X)$ and each $E_\epsilon^c(f)$ meets every element of $E_c(M)$, then $f \in M$.*

*Proof.* *Necessity:* Suppose that $f \notin M$. So $(M, f) = C_c^*(X)$. Hence $h+fg = 1$, for some $h \in M$ and $g \in C_c^*(X)$. Let $u$ be an upper bound for $g$ and $0 < \epsilon < 1$. Then

$$\varnothing = E_\epsilon^c(1) = E_\epsilon^c(h + fg) \supseteq E_{\frac{\epsilon}{2}}^c(h) \cap E_{\frac{\epsilon}{2}}^c(fg) \supseteq E_{\frac{\epsilon}{2}}^c(h) \cap E_{\frac{\epsilon}{2u}}^c(f),$$

which contradicts with our assumption, since $E_{\frac{\epsilon}{2}}^c(h) \cap E_{\frac{\epsilon}{2u}}^c(f) \neq \varnothing$. So we are done.

*Sufficiency:* Let $M'$ be a maximal ideal of $C_c^*(X)$ containing $M$ and $f \in M'$. Then $E_c(M) \subseteq E_c(M')$ and $E_\epsilon^c(f) \in E_c(M')$, for all $\epsilon$. Since $E_c(M')$ is an $e_c$-filter, $E_\epsilon^c(f)$ meets every element of $E_c(M')$. Hence it also meets each element of $E_c(M)$. Now, by hypothesis $f \in M$. Therefore $M = M'$, which gives the result. □

## 3. Uniform norm topology on $C_c^*(X)$ and related closed ideals

Consider the supremum-norm on $C^*(X)$, i.e., $\|f\| = \sup_{x\in X} |f(x)|$, where $f \in C^*(X)$. So its restriction on $C_c^*(X)$ is also the supremum-norm. This defines a metric $d$ as usual, $d(f, g) = \|f - g\|$. The resulting metric topology is called the uniform norm topology on $C_c^*(X)$. Convergence in this topology is uniform convergence of the functions. A base for the neighborhood system at $g$ consists of all sets of the form

$$\{f : \|f - g\| \leq \epsilon\} \qquad (\epsilon > 0).$$

Equivalently, a base at $g$ is given by all sets

$$\{f : |f(x) - g(x)| \leq u(x) \text{ for every } x \in X\},$$

where $u$ is a positive unit of $C_c^*(X)$.

If $I$ is an ideal in $C_c^*(X)$ then its closure in $C_c^*(X)$ is denoted by cl$I$.

**Proposition 3.1.** *Let $I$ be an ideal in $C_c^*(X)$. Then:*

(i) *cl$I$ is ideal.*
(ii) *If $I$ is a proper ideal then cl$I$ is also a proper ideal.*
(iii) *If $I$ is an $e_c$-ideal then it is a closed ideal.*

*Proof.* (i). Let $f, g \in$ cl$I$, $h \in C_c^*(X)$ and let $u$ be an upper bound for $h$ and $\epsilon > 0$ is fixed. Then for some $f' \in N_{\frac{\epsilon}{2}}(f) \cap I$ and $g' \in N_{\frac{\epsilon}{2}}(g) \cap I$ we have $f' + g' \in N_\epsilon(f + g) \cap I$. Moreover, there exists $f_1 \in N_{\frac{\epsilon}{u}}(f) \cap I$ and hence $f_1h \in N_\epsilon(fh) \cap I$. So cl$I$ contains $f + g$ and $fh$. Hence cl$I$ is ideal. (ii). If cl$I$ is not a proper ideal then $1 \in$ cl$I$ and hence $N\epsilon(1) \cap I$ contains a unit element of $C_c^*(X)$ such as $f$, since $1 - \epsilon < f < 1 + \epsilon$ gives $f$ is bounded away from zero (of course, when $0 < \epsilon < 1$). But this is impossible since $f \in I$. Thus cl$I$ is a proper ideal.
(iii). Let $g \in$ cl$I$ and $\epsilon > 0$ arbitrary. Then for some $f \in N_{\frac{\epsilon}{2}}(g) \cap I$ and all $x \in E_{\frac{\epsilon}{2}}^c(f)$, we have

$$|g(x)| = |g(x) - f(x) + f(x)| \leq |g(x) - f(x)| + |f(x)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $E^c_{\frac{\epsilon}{2}}(f) \subseteq E^c_\epsilon(g)$. Since the $z_c$-filter $E_c(I)$ contains $E^c_{\frac{\epsilon}{2}}(f)$, it also contains $E^c_\epsilon(g)$, for all $\epsilon$. So $g \in E_c^{-1}(E_c(I)) = I$ and therefore cl$I \subseteq I$. This proves that $I$ is closed and hence the proof is complete. □

Immediately, we find there is no proper dense ideal in $C^*_c(X)$, and further maximal ideals of $C^*_c(X)$ and hence every intersection of them are closed, see Corollary 2.8 and (iii) of the above proposition.

We recall that [6, 1D(1)] plays a useful role in the context of $C(X)$. The following is the counterpart for $C_c(X)$.

**Proposition 3.2.** *If $f, g \in C_c(X)$ and $Z(f)$ is a neighborhood of $Z(g)$, then $f = gh$ for some $h \in C_c(X)$.*

In the remainder of this section, the zero-dimensional topological space $X$ will be assumed to be compact. Hence it is $c$-pseudo-compact, i.e, $C_c(X) = C^*_c(X)$.

**Lemma 3.3.** *Let $X$ be a compact space, $I$ an ideal in $C_c(X)$, $f \in C_c(X)$ and $Z(f)$ a neighborhood of $\cap Z_c[I]$. Then $f \in I$.*

*Proof.* First, we recall that $X$ is compact if and only if the intersection of members of any collection consisting of nonempty closed subsets of $X$ with the finite intersection property (i.e., the intersection of each of a finite number of them is nonempty) is nonempty. The lemma is obvious when $I = C_c(X)$. Now, if $I$ is a proper ideal in $C_c(X)$ then $Z_c[I]$ satisfies the finite intersection property and hence $\cap Z_c[I] \neq \varnothing$. By assumption $\cap Z_c[I] \subseteq \text{int}Z(f)$. Hence $X \setminus \text{int}Z(f) \subseteq \bigcup_{g\in I} \text{coz}(g)$ and so $X = \bigcup_{g\in I} \text{coz}(g) \cup \text{int}Z(f)$. By compactness of $X$, there are a finite number of elements of $I$, say $g_1, g_2, \ldots, g_n$, such that

$$X = \bigcup_{i=1}^{n} \text{coz}(g_i) \cup \text{int}Z(f).$$

Now, if we let $g = \sum_{i=1}^n g_i^2$ then $g \in I$ and $\varnothing \neq Z(g) = \bigcap_{i=1}^n Z(g_i) \subseteq \text{int}Z(f)$. In view of Proposition 3.2, $f$ is a multiple of $g$ and hence it is contained in $I$. So the proof is complete. □

**Proposition 3.4.** *If $g \in C_c(X)$ and $\epsilon > 0$ is fixed, then there exists $f \in C_c(X)$ such that $\|g - f\| \leq \epsilon$ and $Z(f)$ is a neighborhood of $Z(g)$.*

*Proof.* The trivial solution is $f = g$, of course when $Z(g)$ is open. In general, it suffices to define

$$f(x) = \begin{cases} g(x) - \epsilon, & \text{if } x \in g^{-1}([\epsilon, +\infty)) \\ 0, & \text{if } x \in E^c_\epsilon(g) \\ g(x) + \epsilon, & \text{if } x \in g^{-1}((-\infty, -\epsilon]). \end{cases}$$

We note that $X$ is the union of three closed sets $g^{-1}([\epsilon, +\infty))$, $E^c_\epsilon(g)$ and $g^{-1}((-\infty, -\epsilon])$ and further $f$ is continuous on each of them. Therefore $f$ is continuous on $X$, i.e., $f \in C(X)$. Notice that the definition of $f$ makes the cardinality of the range of $f$ the same cardinality of the range of $g$. Hence this leads us $f \in C_c(X)$. Moreover, $\|g - f\| \leq \epsilon$. Evidently, $Z(g) \subseteq g^{-1}((-\epsilon, \epsilon)) \subseteq \text{int}Z(f)$ which yields $Z(f)$ is a neighborhood of $Z(g)$. □

**Theorem 3.5.** *Let $I$ be a proper ideal in $C_c(X)$, $\overline{I} = \cap\{M_p^c : M_p^c \supseteq I\}$ and $J = \{g \in C_c(X) : Z(g) \supseteq \cap Z_c[I]\}$. Then:*

(i) $\overline{I} = J$.

(ii) $\cap Z_c[I] = \cap Z_c[\overline{I}]$.

*Proof.* (i). Let $g \in J$ and $M_p^c$ be a fixed maximal ideal of $C_c(X)$ containing $I$. Then $Z(g) \supseteq \cap Z_c[I] \supseteq \cap Z_c[M_p^c] = \{p\}$. So $g(p) = 0$ and hence $g \in M_p^c$. Therefore $g \in \overline{I}$. For the reverse inclusion, we show that if $g \notin J$ then $g \notin \overline{I}$. If $g \notin J$ then there exists $x \in \cap Z_c[I] \setminus Z(g)$. So $I \subseteq M_x^c$ but $g \notin M_x^c$. This means that $g \notin \overline{I}$. The proof of (i) is now complete.

(ii). By (i), we have $\cap Z_c[\overline{I}] = \cap Z_c[J] \supseteq \cap Z_c[I]$. On the other hand, $I \subseteq \overline{I}$ implies $Z_c[I] \subseteq Z_c[\overline{I}]$ and therefore $\cap Z_c[I] \supseteq \cap Z_c[\overline{I}]$. So it gives the result. □

**Corollary 3.6.** *Let $I$ be a proper ideal in $C_c(X)$ and $\overline{I}$ as defined in Theorem 3.5. Then $\overline{I} = clI$.*

*Proof.* Since maximal ideals are closed, $\bigcap_{I \subseteq M} M$ is also closed, where $M$ is a maximal ideal in $C_c(X)$. Therefore $\text{cl}I \subseteq \bigcap_{I \subseteq M} M \subseteq \overline{I}$. Let $g \in \overline{I}$ and $N_\epsilon(g)$ is a neighborhood of $g$. By Proposition 3.4, there is $f$ such that $Z(f)$ is a neighborhood of $Z(g)$ and $\|g - f\| \leq \epsilon$. Hence, by Theorem 3.5, $\cap Z_c[I] \subseteq Z(g) \subseteq \text{int}Z(f)$ and therefore Lemma 3.3 implies $f \in I$. Now, since $f \in N_\epsilon(g) \cap I$, it gives $g \in \text{cl}I$. So $\overline{I} \subseteq \text{cl}I$ and we are done. □

We conclude the article with the following results for the proper ideals of $C_c(X)$. Corollary 3.7 is a consequence of Corollary 2.8, Proposition 3.1 (iii) and Corollary 3.6; by the same results, plus Corollary 3.7 we obtain Corollary 3.8; finally Corollary 3.9 is the combination of Corollaries 3.7 and 3.8.

**Corollary 3.7.** *An ideal $I$ of $C_c(X)$ is closed in $C_c(X)$ if and only if it is an intersection of maximal ideals of $C_c(X)$.*

**Corollary 3.8.** *An ideal $I$ of $C_c(X)$ is an $e_c$-ideal if and only if it is closed in $C_c(X)$.*

**Corollary 3.9.** *An ideal $I$ of $C_c(X)$ is an $e_c$-ideal if and only if it is an intersection of maximal ideals of $C_c(X)$.*

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