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Some aspects of Isbell-convex quasi-metric spaces

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Abstract

We introduce and investigate the concept of geodesic bicombing in T_0-quasi-metric spaces. We prove that any Isbell-convex T_0-quasi-metric space admits a geodesic bicombing which satisfies the equivariance property. Additionally, we show that many results on geodesic bicombing hold in quasi-metric settings, provided that nonsymmetry in quasi-metric spaces holds.

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1. Introduction

Let (X, d) be a metric space. A map $\sigma : X \times X \times [0,1] \to X$ is said to be a geodesic bicombing if for every $(x, y) \in X \times X$,

$$\sigma(x, y, 0) = x, \ \ \sigma(x, y, 1) = y \tag{1.1}$$

and

$$d(\sigma(x, y, t), \sigma(x, y, t')) = |t - t'| d(x, y) \tag{1.2}$$

whenever $t, t' \in [0, 1]$ (see [4, 10]). Furthermore, a geodesic bicombing σ on a metric space (X, d) is conical if

$$d(\sigma(x, y, t), \sigma(x', y', t)) \leq (1 - t)d(x, x') + td(y, y')$$

whenever $t, t' \in [0, 1]$ and $x, y, x', y' \in X$.

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In [4] Descombes and Lang discussed and compared three different convexity notions (convex and consistent, convex and conical) for geodesic bicombings. They proved that Busemann spaces, and in particular CAT(0) spaces admit geodesic bicomings which are convex and consistent. Additional to this, they proved that every Gromov hyperbolic group acts geometrically on a proper finite-dimensional metric space with convex and consistent geodesic bicombing.

Isbell [6] and Dress [5] developed independently the concept of injective hull in the category of metric spaces with nonexpansive maps as morphisms. Isbell proved that the injective hull (unique up to isometry) of a metric space is hyperconvex by appealing to Zorn's lemma. Later on Lang [10] presented a new proof which is more constructive of the same result. Furthermore, Lang proved that every metric space which is injective, admits a conical geodesic bicombing.

The injective hull construction for metric spaces has been generalized in the category of T_0-quasi-metric spaces with nonexpansive maps as morphisms (see [7]). Furthermore, an explicit description of the algebraic and vector lattice operations on the Isbell-convex hull of an asymmetrically normed linear vector space is proved in [3]. Naturally this led to the speculation that the Isbell-convex hull of an Isbell-convex T_0-quasi-metric space admits a conical geodesic bicombing. The aim of this article is to give a careful and complete proof of the aforementioned speculation.

We also discuss the continuity of a geodesic bicombing on a T_0-quasi-metric space. Furthermore, we prove that for a conical geodesic bicombing σ on a T_0-quasi-metric space (X, q), if a set A is bounded σ-convex on the set $\mathcal{P}_0(X)$ of nonempty subsets of (X, q), then its double closure $cl_{\tau(q)}A \cap cl_{\tau(q^{-1})}A$ is also bounded σ-convex on $\mathcal{P}_0(X)$. Let us mention that a conical geodesic bicombing on a T_0-quasi-metric space enjoys some property with the Takahashi convex structure on the same T_0-quasi-metric space. For details on Takahashi convex structures on a T_0-quasi-metric space, we refer the reader to [9].

2. Preliminaries

We start by recalling some useful concepts that we are going to use in the sequel.

Definition 2.1. Let X be a nonempty set and $q : X \times X \to [0, \infty)$ be a map. Then q is a *quasi-pseudometric* on X if

(a) $q(x, x) = 0$ whenever $x \in X$, and

(b) $q(x, z) \leq q(x, y) + q(y, z)$ whenever $x, y, z \in X$.

If q is a quasi-pseudometric on a set X, then the pair (X, q) is called a *a quasi-pseudometric space*. Moreover, we say that q is a T_0-quasi-metric provided that it satisfies the additional condition that for any $x, y \in X$, $q(x, y) = 0 = q(y, x)$ implies that $x = y$. The set X together with a T_0-quasi-metric on X is called a *quasi-metric space*.

Remark 2.2. Note that if q is a quasi-metric on X, then $q^{-1} : X \times X \to [0,\infty)$ defined by $q^{-1}(x,y) = q(y,x)$ whenever $x,y \in X$ is also a quasi-pseudometric on X, called the *conjugate quasi-pseudometric* of q. As usual, a quasi-pseudometric d on X such that $q = q^{-1}$ is called a *pseudometric* on X. Furthermore, the map $q^s = \max\{q, q^{-1}\}$ is a pseudometric on X. If q is a T_0-quasi-metric on X, then q^s is a metric on X.

Let (X,q) be a quasi-pseudometric space and for each $x \in X$ and $r \in [0,\infty)$, let $C_q(x,r) = \{y \in X : q(x,y) \leq r\}$ be the $\tau(q^{-1})$-closed ball of centre x and radius r. Furthermore, the *open ball* with centre x and radius r is represented by $B_q(x,r) = \{y \in X : q(x,y) < r\}$.

Example 2.3. Let (X,q) be a T_0-quasi-metric space. For any $x,y \in X$ with $x \neq y$ and $q(x,y) + q(y,x) \neq 0$, the function $u_{q(x,y),q(y,x)} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$u_{q(x,y),q(y,x)}(\lambda, \lambda^{'}) = \begin{cases} (\lambda - \lambda^{'})q(x,y) & \text{if} \quad \lambda \geq \lambda^{'} \\ (\lambda^{'} - \lambda)q(y,x) & \text{if} \quad \lambda < \lambda^{'} \end{cases}$$

is a T_0-quasi-metric.

Take any T_0-quasi-metric space (X,q). If $q(x,y) = 1$ and $q(y,x) = 0$ whenever $x,y \in X$, then the T_0-quasi-metric $u_{q(x,y),q(y,x)}$ is the standard T_0-quasi-metric u on $\mathbb{R}$, where $u(x,y) = \max\{0, x-y\} = x \dot{-} y$ whenever $x,y \in \mathbb{R}$.

Consider a T_0-quasi-metric space (X,q). Let $\mathcal{P}_0(X)$ be the set of all nonempty subsets of X. We recall that for any given $P \in \mathcal{P}_0(X)$, $q(P,x) = \inf\{q(p,x) : p \in P\}$ and $q(x,P) = \inf\{q(x,p) : p \in P\}$ for all $x \in X$.

For any $P, Q \in \mathcal{P}_0(X)$, the so-called *Hausdorff (-Bourbaki) quasi-pseudometric* q_H on $\mathcal{P}_0(X)$ is defined by

$$q_H(P,Q) = \sup_{x \in Q} q(P,x) \vee \sup_{x \in P} q(x,Q).$$

It is well-known that q_H is an extended quasi-pseudometric (q_H may attain the value ∞, then the triangle inequality is interpreted in the obvious way). Moreover, q_H is a T_0-quasi-metric if we restrict the set $\mathcal{P}_0(X)$ to the nonempty subsets of P of X which satisfy $P = cl_{\tau(q)}P \cap cl_{\tau(q^{-1})}P$ (see [2, 8]).

Definition 2.4. ([9, Definition 7]) Let (X,q) be a T_0-quasi-metric space. For any subset P of X, we call $cl_{\tau(q)}P \cap cl_{\tau(q^{-1})}P$ the *double closure* of P. Moreover if $P = cl_{\tau(q)}P \cap cl_{\tau(q^{-1})}P$, we say that P is *doubly closed*.

Definition 2.5. ([7, Definition 2]) A quasi-pseudometric space (X,q) is called *Isbell-convex* (or *q-hyperconvex*) provided that for any family $(x_i)_{i \in I}$ of points in X and families $(r_i)_{i \in I}$ and $(s_i)_{i \in I}$of nonnegative real numbers satisfying $q(x_i,x_j) \leq r_i + s_j$ whenever $i,j \in I$, the following condition hold:

$$\bigcap_{i \in I} [C_q(x_i,r_i) \cap C_{q^{-1}}(x_i,s_i)] \neq \varnothing.$$

For more details about the theory of Isbell-convex T_0-quasi-metric spaces, we refer the reader to [3, 7, 11].

3. Geodesic bicombing

We start this section with the following observation.

Remark 3.1. We observed that the condition (1.2) is unsuitable for a T_0-quasi-metric space (X, d) that is not a metric: Indeed if d is a T_0-quasi-metric space with properties (1.1) and (1.2), then it satisfies

$$|0-1|d(x,y) = d(\sigma(x,y,0), \sigma(x,y,1)) = d(\sigma(y,x,1), \sigma(y,x,0)) = |1-0|d(y,x)$$

whenever $x, y \in X$ and thus d would be a metric. Therefore, for a T_0-quasi-metric space (X, d) we propose the condition (1.2) differently in Definition 3.2.

Let $I = [0, 1]$ be the set of real unit interval. Using a different terminology, it was essentially observed by Remark 3.1 that in a T_0-quasi-metric space the concept of geodesic bicombing can be modified in the following way:

Definition 3.2. Let (X, q) be a T_0-quasi-metric space. A *geodesic bicombing* σ on (X, q) is a map $\sigma : X \times X \times I \to X$ such that for each $(x, y) \in X \times X$, $\sigma(x, y, 0) = x$, $\sigma(x, y, 1) = y$ and

$$q(\sigma(x,y,\lambda), \sigma(x,y,\lambda')) = (\lambda' - \lambda)q(x,y) \text{ if } \lambda' \geq \lambda \tag{3.1}$$

and

$$q(\sigma(x,y,\lambda), \sigma(x,y,\lambda')) = (\lambda - \lambda')q(y,x)) \text{ if } \lambda' < \lambda \tag{3.2}$$

whenever $\lambda, \lambda' \in I$.

Definition 3.3. Let σ be a geodesic bicombing on a T_0-quasi-metric space (X, q). We say that σ satisfies the *equivariance property* if

$$\sigma(x,y,\lambda) = \sigma(y,x,1-\lambda) \text{ whenever } x, y \in X \text{ and } \lambda \in [0,1].$$

Lemma 3.4. *If σ is a geodesic bicombing on a T_0-quasi-metric space (X, q), then σ is a geodesic bicombing on the conjugate T_0-quasi-metric space (X, q^{-1}). Furthermore, σ is a geodesic bicombing on the metric space (X, q^s)*

Proof. Suppose that σ is a geodesic bicombing on (X, q). Let $x, y \in X$ and $\lambda, \lambda' \in I$. Obviously, $\sigma(x, y, 0) = x$, $\sigma(x, y, 1) = y$.

If $\lambda' \geq \lambda$, then

$$q^{-1}(\sigma(x,y,\lambda), \sigma(x,y,\lambda')) = q(\sigma(x,y,\lambda'), \sigma(x,y,\lambda)) = (\lambda' - \lambda)q^{-1}(x,y).$$

If $\lambda > \lambda'$, we have

$$q^{-1}(\sigma(x,y,\lambda), \sigma(x,y,\lambda') = q(\sigma(x,y,\lambda'), \sigma(x,y,\lambda)) = (\lambda - \lambda)q^{-1}(y,x).$$

So σ is a geodesic bicombing on (X, q^{-1}).

If $\lambda' \geq \lambda$ for $\lambda, \lambda' \in I$, then we have

$$\begin{aligned} q^s(\sigma(x,y,\lambda), \sigma(x,y,\lambda')) &= \max\{q(\sigma(x,y,\lambda), \sigma(x,y,\lambda')), q^{-1}(\sigma(x,y,\lambda), \sigma(x,y,\lambda'))\} \\ &= \max\{(\lambda' - \lambda)q(x,y), (\lambda' - \lambda)q^{-1}(x,y)\} = (\lambda' - \lambda)q^s(x,y). \end{aligned}$$

Similarly if $\lambda > \lambda'$,

$$q^s(\sigma(x,y,\lambda), \sigma(x,y,\lambda')) = (\lambda - \lambda')q^s(x,y).$$

Hence $q^s(\sigma(x,y,\lambda),\sigma(x,y,\lambda')) = |\lambda-\lambda'|q^s(x,y)$ whenever $\lambda,\lambda' \in I$. $\square$

Example 3.5. If we equip $(\mathbb{R},u)$ with $\sigma(x,y,\lambda) = (1-\lambda)x+\lambda y$ whenever $x,y \in \mathbb{R}$ and $\lambda \in I$, then σ is a geodesic bicombing on $(\mathbb{R},u)$ and σ is called the standard geodesic bicombing on $(\mathbb{R},u)$.

Indeed, if $\lambda' \geq \lambda$, we have

$$\begin{aligned} u(\sigma(x,y,\lambda),\sigma(x,y,\lambda')) &= u((1-\lambda)x+\lambda y,(1-\lambda')x+\lambda' y) \\ &= \max\{0,(1-\lambda)x+\lambda y-[(1-\lambda')x+\lambda' y]\}. \end{aligned}$$

So

$$u(\sigma(x,y,\lambda),\sigma(x,y,\lambda')) = \max\{0,(\lambda'-\lambda)x-(\lambda'-\lambda)y\} = (\lambda'-\lambda)u(x,y).$$

By similar arguments if $\lambda > \lambda'$, then

$$u(\sigma(x,y,\lambda),\sigma(x,y,\lambda')) = (\lambda-\lambda')u(y,x).$$

Lemma 3.6. *Let σ be a geodesic bicombing on a T_0-quasi-metric space (X,q). Then the map σ^{-1} defined by $\sigma^{-1}(x,y,\lambda) = \sigma(y,x,1-\lambda)$ whenever $x,y \in X$ and $\lambda \in I$, is a geodesic bicombing on (X,q). The geodesic bicombing σ^{-1} is called* reversible *geodesic bicombing of σ (see* [4, p.2]*).*

Proof. Let $x,y \in X$ and $\lambda,\lambda' \in I$. We have $\sigma^{-1}(x,y,0) = \sigma(y,x,1) = x$ and $\sigma^{-1}(x,y,1) = \sigma(y,x,0) = y$.

If $\lambda' \geq \lambda$, then $(1-\lambda) > (1-\lambda')$. It follows that

$$q(\sigma^{-1}(x,y,\lambda),\sigma^{-1}(x,y,\lambda')) = q(\sigma(y,x,1-\lambda),\sigma(y,x,1-\lambda')) = (\lambda'-\lambda)q(y,x).$$

If $\lambda > \lambda'$, then one sees that

$$q(\sigma^{-1}(x,y,\lambda),\sigma^{-1}(x,y,\lambda')) = (\lambda-\lambda')q(x,y).$$

$\square$

Example 3.7. Let C be a convex subset of a real linear space X equipped with the asymmetric norm $\|.|$. Then $\sigma(x,y,\lambda) = (1-\lambda)x+\lambda y$ whenever $x,y \in C$ and $\lambda \in I$, is a geodesic bicombing on (C,d), where $d(x,y) = \|x-y|$ whenever $x,y \in C$.

Lemma 3.8. *Let (X,q) be a T_0-quasi-metric space with a geodesic bicombing σ. Then we have $\sigma(x,x,\lambda) = x$ whenever $x \in X$ and $\lambda \in I$.*

Proof. Let $x \in X$ and $\lambda \in I$. Then

$$q(x,\sigma(x,x,\lambda)) \leq q(x,\sigma(x,x,0))+q(\sigma(x,x,0),\sigma(x,x,\lambda)) = q(x,x)+\lambda q(x,x) = 0.$$

Thus $q(x,\sigma(x,x,\lambda)) = 0$. Furthermore,

$$q(q(\sigma(x,x,\lambda),x) \leq q(\sigma(x,x,\lambda),\sigma(x,x,0))+q(\sigma(x,x,0),x) = \lambda q(x,x)+q(x,x) = 0.$$

Hence $q(\sigma(x,x,\lambda),x) = 0 = q(x,\sigma(x,x,\lambda))$. We have $\sigma(x,x,\lambda) = x$ by T_0-property of (X,q). $\square$

Note that a geodesic bicombing need not to be unique. To obtain the following embedding we assume that the geodesic bicombing is unique.

Lemma 3.9 (compare [9, Proposition 4]). *Suppose that σ is the unique geodesic bicombing on the T_0-quasi-metric space (X,q). If for every $x,y \in X$ with $x \neq y$, the map $\psi : (I, u_{q(x,y),q(y,x)}) \to (X,q)$ defined by $\psi(\lambda) = \sigma(x,y,\lambda)$ whenever $\lambda \in I$ is an isometric embedding. (Here $u_{q(x,y),q(y,x)}$ is the restriction of the T_0-quasi-metric $u_{q(x,y),q(y,x)}$ given in Example 2.3 to I).*

Proof. For $\lambda, \lambda' \in I$, we show that

$$q(\sigma(x,y,\lambda), \sigma(x,y,\lambda')) = u_{q(x,y),q(y,x)}(\lambda,\lambda').$$

If $\lambda' \geq \lambda$, then $q(\sigma(x,y,\lambda),\sigma(x,y,\lambda')) = (\lambda'-\lambda)q(x,y) = u_{q(x,y),q(y,x)}(\lambda,\lambda')$.
If $\lambda > \lambda'$, then $q(\sigma(x,y,\lambda),\sigma(x,y,\lambda')) = (\lambda-\lambda')q(y,x) = u_{q(x,y),q(y,x)}(\lambda,\lambda')$. ☐

Lemma 3.10. *Let σ be a geodesic bicombing on a T_0-quasi-metric space (X,q). Then whenever $x,y,u \in X$ and $\lambda \in I$, σ satisfies the following inequalities:*

$$q(u,\sigma(x,y,\lambda)) \leq q(u,x) + \lambda q(x,y) \tag{3.3}$$

$$q(u,\sigma(x,y,\lambda)) \leq q(u,y) + (1-\lambda)q(u,x) \tag{3.4}$$

$$q(\sigma(x,y,\lambda),u) \leq \lambda q(y,x) + q(x,u) \tag{3.5}$$

$$q(\sigma(x,y,\lambda),u) \leq (1-\lambda)q(x,y) + q(u,y). \tag{3.6}$$

Proof. We prove only (3.3) and (3.6), then (3.4) and (3.5) follow analogously. Let $x,y,x',y' \in X$ and $\lambda \in I$.

Then $q(u,\sigma(x,y,\lambda)) \leq q(u,\sigma(x,y,0)) + q(\sigma(x,y,0),\sigma(x,y,\lambda))$ by triangle inequality. Since $\sigma(x,y,0) = x$ and from the equality (3.1) ($\lambda \geq 0$), it follows that

$$q(u,\sigma(x,y,\lambda)) \leq q(u,x) + \lambda q(x,y).$$

Furthermore, $q(\sigma(x,y,\lambda),u) \leq q(\sigma(x,y,\lambda),\sigma(x,y,1)) + q(\sigma(x,y,1),u)$, then

$$q(\sigma(x,y,\lambda),u) \leq (1-\lambda)q(x,y) + q(y,u)$$

by $\sigma(x,y,1) = y$ and from the equality (3.1) ($1 \geq \lambda$). ☐

Proposition 3.11 (compare [9, Proposition 2]). *Let σ be a geodesic bicombing on a T_0-quasi-metric space (X,q). Then for each $x \in X$ and $\lambda \in I$, σ is continuous at (x,x,λ), where X carries the topology $\tau(q)$ (or $\tau(q^{-1})$).*

Proof. Consider the convergent sequence $((x_n,y_n,\lambda_n))$ in $X \times X \times I$. Suppose that $((x_n,y_n,\lambda_n))$ converges to (x,x,λ) with respect to the topology induced by q on X. The topology on I does not really matter. We have to prove that the sequence $(\sigma(x_n,y_n,\lambda_n))$ converges to $(\sigma(x,x,\lambda))$. From Lemma 3.8, we know that $\sigma(x, x, \lambda) = x$ whenever $\lambda \in I$. By inequality (3.4) of Lemma 3.10, whenever $n \in \mathbb{N}$ we have

$$q(x, \sigma(x_n, y_n, \lambda_n)) \leq q(x, y_n) + (1 - \lambda_n) q(x, x_n).$$

Therefore, the sequence $(\sigma(x_n, y_n, \lambda_n))$ converges to $(\sigma(x, x, \lambda))$. One obtains the similar result by using the topology induced by q^{-1} on X and inequality (3.3). □

4. Conical geodesic bicombing

Definition 4.1. Let σ be a geodesic bicombing on a T_0-quasi-metric space (X, q). Then σ is said to be *conical* if

$$q(\sigma(x, y, \lambda), \sigma(x', y', \lambda)) \leq (1 - \lambda) q(x, x') + \lambda q(y, y') \tag{4.1}$$

whenever $x, y, x', y' \in X$ and $\lambda \in I$. Furthermore, the geodesic bicombing σ is called *convex* if the function $\lambda \mapsto q(\sigma(x, y, \lambda), \sigma(x', y', \lambda))$ is convex on I whenever $x, y, x', y' \in X$ and $\lambda \in I$.

The following ideas are not new and were inspired from [9].

Let σ be a conical geodesic bicombing on a T_0-quasi-metric space (X, q). A subset C of X is called *σ-convex* provided that $\sigma(c, c', \lambda) \in C$ whenever $c, c' \in C$ and $\lambda \in I$. Observe that X is σ-convex subset of itself. Moreover, each σ-convex subset C of X carries a natural conical geodesic bicombing, which is the restriction of σ to $C \times C \times I$.

Proposition 4.2. *Let σ be a conical geodesic bicombing on a T_0-quasi-metric space (X, q). Then whenever $x \in X$ and $r > 0$, the closed balls $C_q(x, r)$ and $C_{q^{-1}}(x, s)$ and the open balls $B_q(x, r)$ and $B_{q^{-1}}(x, s)$ are σ-convex subsets of X.*

Proof. We only prove that $C_q(x, r)$ is σ-convex, the proofs related to the balls $C_{q^{-1}}(x, s)$, $B_q(x, r)$ and $B_{q^{-1}}(x, s)$ follow analogously. Suppose that $x \in X$, $r > 0$ and $\lambda \in I$. Let $y, z \in C_q(x, r)$. Then

$$q(x, \sigma(y, z, \lambda)) = q(\sigma(x, x, \lambda), \sigma(y, z, \lambda))$$

by Lemma 3.8. Furthermore,

$$q(x, \sigma(y, z, \lambda)) \leq (1 - \lambda) q(x, y) + \lambda q(x, z) \leq (1 - \lambda) r + \lambda r = r$$

since σ is conical, $q(x, y) \leq r$ and $q(x, z) \leq r$. Thus $\sigma(y, z, \lambda) \in C_q(x, r)$. □

Obviously, one can prove that the intersection of any family of σ-convex subsets of (X, q) is σ-convex too.

Let $\mathcal{CB}_0(X)$ be the subcollection of bounded σ-convex elements of $\mathcal{P}_0(X)$. In this case q_H is a quasi-pseudometric since $q_H(A, B) < \infty$. For more details about how $q_H(A, B) < \infty$, we refer the reader to [9, p.13].

Let σ be a conical geodesic bicombing on a T_0-quasi-metric space (X, q). For any $A, B \in \mathcal{CB}_0(X)$ and $\lambda \in I$, set $\sigma(A, B, \lambda) := \{\sigma(a, b, \lambda) : a \in A, b \in B\}$.

Then we have that $\sigma(A,B,\lambda)$ is nonempty and bounded. Let $a,a'\in A$ and $b,b'\in B$. Then we have that

$$q(\sigma(a,b,\lambda),\sigma(a',b',\lambda))\leq(1-\lambda)q(a,a')+\lambda q(b,b')\leq(1-\lambda)\mathrm{diam}(A)+\lambda\mathrm{diam}(B)$$

since σ is conical. One can add some conditions on σ in order the set $\sigma(A,B,\lambda)$ preserves σ-convexity.

We leave the proof of the following lemma to the reader.

Lemma 4.3 (compare [9, Lemma 3])**.** *Let σ be a conical geodesic bicombing on a T_0-quasi-metric space (X,q). If $A\in\mathcal{CB}_0(X)$, then its double closure $cl_{\tau(q)}A\cap cl_{\tau(q^{-1})}A$ is contained in $\mathcal{CB}_0(X)$.*

5. Injective spaces

We start by recalling the construction of Isbell-convex hull (injective hull) of a T_0-quasi-metric space (X,q) and we refer the reader to [7] for more details.

Let $\mathcal{FP}(X,q)$ be the set of all pairs of functions $f=(f_1,f_2)$, where $f_i: X\to[0,\infty)(i=1,2)$. A function pair (f_1,f_2) is said to be ample on (X,q) if $q(x,y)\leq f_2(x)+f_1(y)$ whenever $x,y\in X$. The set of all function pairs which are ample on (X,q) will be denoted by $\mathcal{A}(X,q)$. For $a\in X$, it is easy to see that the function pair $f_a(x)=(q(a,x),q(x,a))$ is element of $\mathcal{A}(X,q)$. For each $(f_1,f_2),(g_1,g_2)\in\mathcal{A}(X,q)$, the map D defined by

$$D((f_1,f_2),(g_1,g_2))=\sup_{x\in X}(f_1(x)\dot{-}g_1(x))\vee\sup_{x\in X}(g_2(x)\dot{-}f_2(x))$$

is an extended T_0-quasi-metric on $\mathcal{A}(X,q)$.

Let $(f_1,f_2)\in\mathcal{A}(X,q)$. We say that (f_1,f_2) is extremal (or minimal) on (X,q) if take $(g_1,g_2)\in\mathcal{A}(X,q)$ such that $g_1(x)\leq f_1(x)$ and $g_2(x)\leq f_2(x)$ for $x\in X$, then $f_1(x)=g_1(x)$ and $f_2(x)=g_2(x)$.

The Isbell-convex hull or the injective hull of (X,q) is the set $\mathcal{E}(X,q)$ of all extremal function pairs on (X,d).

It is well-known that if a function pair (f_1,f_2) is extremal, then $f_1:(X,q^{-1})\to(\mathbb{R},u)$ and $f_2:(X,q)\to(\mathbb{R},u)$ are nonexpansive map, that is

$$f_1(x)-f_1(y)\leq q(y,x)$$

and

$$f_2(x)-f_2(y)\leq q(x,y)$$

whenever $x,y\in X$. Furthermore,

$$f_1(x)=\sup_{y\in X}(q(y,x)\dot{-}f_2(y))$$

and

$$f_2(x)=\sup_{y\in X}(q(x,y)\dot{-}f_1(y))$$

whenever $x\in X$. For $a\in X$, it is easy to see that

$$f_a(x)=(q(a,x),q(x,a))\in\mathcal{E}(X,q).$$

Let $\mathcal{FPN}_{\exp}(X,\mathbb{R})$ be the set of all function pairs whose the first component is nonexpansive on (X,q^{-1}) and the second component is nonexpansive on (X,q). Observe that if $(f_1,f_2)\in\mathcal{E}(X,q)$, then $(f_1,f_2)\in\mathcal{FPN}_{\exp}(X,\mathbb{R})$.

We now consider the set $\mathcal{A}_1(X,q):=\mathcal{A}(X,q)\cap\mathcal{FPN}_{\exp}(X,\mathbb{R})$.

Lemma 5.1. *Let (X,q) be a T_0-quasi-metric space. If we equip $\mathcal{FPN}_{exp}(X,\mathbb{R})$ with the restriction of the extended T_0-quasi-metric D, then $D((f_1,f_2),(g_1,g_2))<\infty$. (Moreover D is a T_0-quasi-metric).*

Proof. Let $(f_1,f_2)\in\mathcal{FPN}_{\exp}(X,\mathbb{R})$. Then $q(x,y)\le f_2(x)+f_1(y)$ whenever $x,y\in X$ and so

$$\sup_{x\in X}(q(x,y)\dot{-}f_2(x))\le f_1(y).$$

Moreover, we have

$$\sup_{x\in X}(f_1(x)\dot{-}q(y,x))\le f_1(y)$$

whenever $x,y\in X$, since f_1 is nonexpansive on (X,q^{-1}).

Thus $D((f_1,f_2),((f_y)_1,(f_y)_2))\le f_1(y)$ whenever $y\in X$. By similar arguments one shows that $D(((f_y)_1,(f_y)_2),(f_1,f_2))\le f_2(y)$ whenever $y\in X$. Therefore, for $y\in X$ we have $D((f_1,f_2),(g_1,g_2))\le f_1(y)+g_2(y)<\infty$ whenever $(f_1,f_2),(g_1,g_2)\in\mathcal{FPN}_{\exp}(X,\mathbb{R})$. $\square$

The following useful result is due to [1]. Its proof is based on Zorn's Lemma but a different proof of Proposition 5.2 can be given without appealing to Zorn's Lemma.

Proposition 5.2. *Let (X,q) be a T_0-quasi-metric space. There exists a retraction map $p:\mathcal{A}(X,q)\to\mathcal{E}(X,q)$, i.e., a map that satisfies the conditions*

- (a) *$D(p((f_1,f_2)),p((g_1,g_2)))\le D((f_1,f_2),(g_1,g_2))$ whenever $(f_1,f_2),(g_1,g_2)\in\mathcal{A}(X,q)$.*
- (b) *$p((f_1,f_2))\le(f_1,f_2)$ whenever $(f_1,f_2)\in\mathcal{A}(X,q)$.(In particular $p((f_1,f_2))=(f_1,f_2)$ whenever $(f_1,f_2)\in\mathcal{E}(X,q)$).*

Remark 5.3. From Proposition 5.2, it follows that if $(f_1,f_2),(g_1,g_2)\in\mathcal{A}(X,q)$, then $D(p((f_1,f_2)),p((g_1,g_2)))$ can be ∞. But if $(f_1,f_2),(g_1,g_2)\in\mathcal{FPN}_{\exp}(X,\mathbb{R})$, then $D(p((f_1,f_2)),p((g_1,g_2)))$ is finite. Therefore, the restriction of the map $p:\mathcal{A}(X,q)\to\mathcal{E}(X,q)$ to $\mathcal{FPN}_{\exp}(X,\mathbb{R})$ is a nonexpansive retraction.

Proposition 5.4. *Let (X,q) be a T_0-quasi-metric space. Then the T_0-quasi-metric spaces $(\mathcal{FPN}_{exp}(X,\mathbb{R}),D)$ and $(\mathcal{E}(X,q),D)$ are injective.*

Proof. Let $\varnothing\ne A\subseteq B\subseteq X$. Consider a map $F:A\to\mathcal{FPN}_{\exp}(X,\mathbb{R})$ defined by $F(a)=f_a=(q(a,.),q(.,a))$ whenever $a\in A$. Obviously F_a is a function pair which is ample, where $q(a,.)$ is nonexpansive on (X,q^{-1}) and $q(.,a)$ is nonexpansive on (X,q).

Let $b\in B$, we set $\overline{f_b}=((\overline{f_b})_1,(\overline{f_b})_2)$ where,

$$(\overline{f_b})_1(x):=\inf_{a\in A}\{(f_a)_1(x)+q(b,a)\}$$

and

$$(\overline{f_b})_2(x) := \inf_{a\in A}\{(f_a)_2(x) + q(a,b)\}$$

whenever $x \in X$.

We have to show that $\overline{f_b} \in \mathcal{FPN}_{\exp}(X,\mathbb{R})$.

For each $x, y \in X$, we have

$$\begin{aligned}(\overline{f_b})_1(x) - (\overline{f_b})_1(y) &= \inf_{a\in A}\{(f_a)_1(x) + q(b,a)\} - \inf_{a'\in A}\{(f_{a'})_1(y) + q(b,a')\}\\ &\leq (f_a)_1(x) + q(b,a) - (f_a)_1(y) - q(b,a) \text{ with } a = a'\\ &\leq q(a,y) + q(y,x) - q(a,y)\\ &= q(y,x).\end{aligned}$$

Similarly,

$$(\overline{f_b})_2(x) - (\overline{f_b})_2(y) \leq q(x,y)$$

whenever $x, y \in X$. So $(\overline{f_b})_1$ and $(\overline{f_b})_2$ are nonexpansive.

To show that the function pair $\overline{f_b}$ is ample, let $x, y \in X$. Then

$$\begin{aligned}(\overline{f_b})_2(x) + (\overline{f_b})_1(y) &\geq \inf_{a,a'\in A}\{(f_a)_2(x) + q(a,b) + (f_{a'})_1(x) + q(b,a')\}\\ &\geq \inf_{a,a'\in A}\{(f_a)_2(x) + (f_{a'})_1(x) + q(a,a')\}\\ &\geq \inf_{a\in A}\{(f_a)_2(x) + q(a,y)\} = \inf_{a\in A}\{q(x,a) + q(a,y)\}\\ &\geq q(x,y).\end{aligned}$$

Therefore, $\overline{f_b} \in \mathcal{FPN}_{\exp}(X,\mathbb{R})$.

Let $b, b' \in B$ and $x \in X$. We show that $D(\overline{f_b}, \overline{f_{b'}}) \leq q(b,b')$.

Indeed,

$$\begin{aligned}(\overline{f_{b'}})_2(x) - q(b,b') &= \inf_{a\in A}\{(f_a)_2(x) + q(a,b')\} - q(b,b')\\ &= \inf_{a\in A}\{(f_a)_2(x) + q(a,b') - q(b,b')\}\\ &\leq \inf_{a\in A}\{(f_a)_2(x) + q(a,b)\}\\ &\leq (\overline{f_b})_2(x).\end{aligned}$$

Hence

$$\sup_{x\in X}\left((\overline{f_{b'}})_2(x) \dot{-} (\overline{f_b})_2(x)\right) \leq q(b,b') \tag{5.1}$$

whenever $b, b' \in B$ and $x \in X$.

Similarly, we have

$$\sup_{x\in X}\left((\overline{f_b})_1(x) \dot{-} (\overline{f_{b'}})_1(x)\right) \leq q(b,b') \tag{5.2}$$

for $b, b' \in B$ and $x \in X$. Combining (5.1) and (5.2) we have

$$D(\overline{f_b}, \overline{f_{b'}}) \leq q(b,b')$$

for $b, b' \in B$. We now show that $\overline{f_b} = f_b$ whenever $b \in A$.

If $b \in A$, then

$$(\overline{f_b})_1(x) \le q(b,x) = (f_b)_1(x) \text{ whenever } x \in X \tag{5.3}$$

since $(\overline{f_b})_1(x) \le q(b,x) + q(b,b)$ and $b \in A$.

Moreover, $(f_b)_1(x) = q(b,x) \le q(b,a) + q(a,x) = (f_a)_1(x) + q(b,a)$ for all $x \in X$. Thus

$$(f_b)_1(x) \le \inf_{a \in A}\{(f_a)_1(x) + q(b,a)\} = (\overline{f_b})_1(x) \tag{5.4}$$

for all $x \in X$ and $b \in B$.

Hence, $(\overline{f_b})_1(x) = (f_b)_1(x)$ for all $x \in X$ and $b \in A$ from (5.3) and (5.4).

Analogously, one shows that $(\overline{f_b})_2(x) = (f_b)_2(x)$ whenever $x \in X$ and $b \in A$. Therefore the map $\overline{F} : B \to \mathcal{FPN}_{\exp}(X,\mathbb{R})$ defined by $\overline{F}(b) = f_b$ whenever $b \in B$, extends F. So $(\mathcal{FPN}_{\exp}(X,\mathbb{R}), D)$ is injective. The injectivity of $(\mathcal{E}(X,q), D)$ follows from Remark 5.3 since $(\mathcal{E}(X,q), D)$ is nonexpansive retract of $(\mathcal{FPN}_{\exp}(X,\mathbb{R}), D)$. $\square$

The following observations are not new since they have been discussed in [10] from the metric point of view.

Remark 5.5. Let (X,q) be a T_0-quasi-metric space.

(a) If $L : (\mathcal{E}(X,q), D) \to (\mathcal{E}(X,q), D)$ is a nonexpansive map that fixes $e_X(X)$ pointwise, then L is an identity on $\mathcal{E}(X,q)$.

Indeed, if $f = (f_1, f_2) \in \mathcal{E}(X,q)$ such that $L(f) = (g_1, g_2)$ for some $g = (g_1, g_2) \in \mathcal{E}(X,q)$, then,

$$\begin{aligned} g_1(x) = D(g, f_x) &= D(L(f), L(f_x)) \\ &\le D(f, f_x) = f_1(x) \quad \text{for } x \in X. \end{aligned}$$

Similarly, $g_2(x) \le f_2(x)$ whenever $x \in X$. By minimality of (f_1, f_2), we have $g_1(x) = f_1(x)$ and $g_2(x) = f_2(x)$ whenever $x \in X$. Hence $f = (f_1, f_2) = (g_1, g_2) = g$. Therefore, $L(f) = f$ whenever $f \in \mathcal{E}(X,q)$.

(b) Since $(\mathcal{E}(X,q), D)$ is injective and e_X is quasi-essential by [11, Remark 16], $(\mathcal{E}(X,q), e_X)$ is an injective hull of (X,q).

(c) If (Y,i) is another injective hull of (X,q), then there exists an isometric embedding of (X,q), and then there exists a unique isometry $I : (\mathcal{E}(X,q), D) \to (Y,i)$ such that $I \circ e_X = i$ by [11, Proposition 10].

In [1], Agyingi et al. proved that if (Y, q_Y) is a T_0-quasi-metric space and X is a subspace of (Y, q_Y), then there exists an isometric embedding $\tau : (\mathcal{E}(X,q), D) \to (Y, q_Y)$ such that $\tau(f)|_X = f$ whenever $f \in \mathcal{E}(X,q)$.

Proposition 5.6. *Let (X,q) be a T_0-quasi-metric space. If $L : (X,q) \to (X,q)$ is an isometry, then there exists a unique isometry $\bar{L} : (\mathcal{E}(X,q), D) \to (\mathcal{E}(X,q), D)$ such that $\bar{L} \circ e_X = e_X \circ L$. Furthermore, $\bar{L}(f) = (f_1 \circ L^{-1}, f_2 \circ L^{-1})$ whenever $f \in \mathcal{E}(X,q)$.*

Proof. Suppose $L : (X,q) \to (X,q)$ is an isometry. Then $e_X \circ L : (X,q) \to (\mathcal{E}(X,q), D)$ is quasi-essential and since $(\mathcal{E}(X,q), D)$ is injective, it follows that $(\mathcal{E}(X,q), e_X \circ L)$ is injective hull of (X,q) by Remark 5.5 (b).

Moreover, by Remark 5.5 (c), there exists a unique isometry such that $\bar{L} \circ e_X = e_X \circ L$.

If $f = (f_1, f_2) \in \mathcal{E}(X, q)$ and $x \in X$, then

$$(\bar{L}(f))_1(x) = D(\bar{L}(f), f_x) = D(\bar{L}(f), \bar{L}(\bar{L}^{-1}(f_x))) = D(f, \bar{L}^{-1}(f_x)).$$

Since $e_X \circ L^{-1} = \bar{L}^{-1} \circ e_X$,

$$f_{L^{-1}(x)} = (e_X \circ L^{-1})(x) = (\bar{L}^{-1} \circ e_X)(x) = \bar{L}^{-1}(f_x),$$

whenever $x \in X$.

Hence

$$(\bar{L}(f))_1(x) = D(f, \bar{L}^{-1}(f_x)) = D(f, f_{L^{-1}(x)}) = f_1(L^{-1}(x)) = (f_1 \circ L^{-1})(x).$$

By similar arguments we have

$$(\bar{L}(f))_2(x) = (f_1 \circ L^{-1})(x)$$

whenever $x \in X$. $\square$

Proposition 5.7. *Let (X, q) be a T_0-quasi-metric space. If $L : (X, q) \to (X, q)$ is an isometry, then the function pair $\psi(f) = \bar{L}(f)$ is ample whenever $f = (f_1, f_2) \in \mathcal{A}(X, q)$. Furthermore, we have $\bar{L}(p(f)) = p(\bar{L}(f))$ whenever $f = (f_1, f_2) \in \mathcal{A}(X, q)$, where p is the map in Proposition 5.2 and $\bar{L}$ is the unique isometry map in Proposition 5.6.*

Proof. Let $f = (f_1, f_2) \in \mathcal{A}(X, q)$. Then for any $x, y \in X$, we have

$$\begin{aligned} (\bar{L}(f))_2(x) + (\bar{L}(f))_1(y) &= (f_2 \circ L^{-1})(x) + (f_1 \circ L^{-1})(y) \\ &= f_2(L^{-1}(x)) + f_1(L^{-1}(y)) \\ &\geq q(L^{-1}(x), L^{-1}(y)) \\ &= q(x, y). \end{aligned}$$

Let $y \in X$. Consider

$$f_1^*(y) = \sup_{x' \in X} \{q(x', y) \dot{-} f_2(x')\},$$

$$f_2^*(y) = \sup_{x' \in X} \{q(y, x') \dot{-} f_1(x')\}$$

and the operator $q(f) = (\frac{1}{2}(f_1 + f_1^*), \frac{1}{2}(f_2 + f_2^*))$ defined in the proof (given in [1]) of Proposition 5.2. Then

$$\begin{aligned} (f_1^* \circ L^{-1})(y) = f_1^*(L^{-1}(y)) &= \sup_{x' \in X} \{q(x', L^{-1}(y)) \dot{-} f_2(x')\} \\ &= \sup_{L^{-1}(L(x')) \in X} \{q(L^{-1}(L(x')), L^{-1}(y)) \dot{-} f_2(L^{-1}(L(x')))\} \\ &= f_1(L^{-1})^*(y) = (f_1 \circ L^{-1})^*(y). \end{aligned}$$

Thus, we have that $(f_1^* \circ L^{-1})(y) = (f_1 \circ L^{-1})^*(y)$ whenever $y \in X$. Hence, whenever $x \in X$ we have

$$\begin{aligned}(q(f)_1 \circ L^{-1})(x) = \left(\frac{1}{2}(f_1 + f_1^*) \circ L^{-1}\right)(x) &= \frac{1}{2}\left(f_1(L^{-1}) + f_1^*(L^{-1})\right)(x)\\ &= \frac{1}{2}\left(f_1 \circ L^{-1} + f_1^* \circ L^{-1}\right)(x)\\ &= q(f \circ L^{-1})_1(x).\end{aligned}$$

Similarly, we can show that $(q(f)_2 \circ L^{-1})(x) = q(f \circ L^{-1})_2(x)$ whenever $x \in X$. Therefore,

$$\bar{L}(p(f))_1 = p(f)_1 \circ L^{-1} = p(f \circ L^{-1})_1 = p(\bar{L}(f))_1$$

and

$$\bar{L}(p(f))_2 = p(f)_2 \circ L^{-1} = p(f \circ L^{-1})_2 = p(\bar{L}(f))_2.$$

□

Proposition 5.8. *Every Isbell-convex T_0-quasi-metric space admits a conical geodesic bicombing which satisfies the equivariance property.*

Proof. Suppose that (X, q) is an Isbell-convex T_0-quasi-metric space. Let $x, y \in X$ and $\lambda \in [0, 1]$, we define a function pair $\varphi_{xy}^{\lambda} = (\varphi_{xy,1}^{\lambda}, \varphi_{xy,2}^{\lambda})$ by

$$\varphi_{xy,1}^{\lambda}(z) = (1-\lambda)(f_x)_1(z) + \lambda(f_y)_1(z)$$

and

$$\varphi_{xy,2}^{\lambda}(z) = (1-\lambda)(f_x)_2(z) + \lambda(f_y)_2(z)$$

whenever $z \in X$. We will prove that $\varphi_{xy}^{\lambda} \in \mathcal{A}_1(X, q)$.

We first show that φ_{xy}^{λ} is ample. Let $z, z' \in X$, then

$$\begin{aligned}\varphi_{xy,2}^{\lambda}(z) + \varphi_{xy,1}^{\lambda}(z') &= (1-\lambda)q(z,x) + \lambda q(z,y) + (1-\lambda)q(x,z') + \lambda q(y,z')\\ &= (1-\lambda)[q(z,x) + q(x,z')] + \lambda[q(z,y) + q(y,z')]\\ &\geq (1-\lambda)q(z,z') + \lambda q(z,z')\\ &= q(z,z').\end{aligned}$$

We now show that $\varphi_{xy,2}^{\lambda}$ is a nonexpansive map on (X, q) and the proof of the fact that $\varphi_{xy,1}^{\lambda}$ is a nonexpansive map on (X, q^{-1}) follow by similar arguments.

Let $z, z' \in X$, then

$$\begin{aligned}\varphi_{xy,2}^{\lambda}(z) - \varphi_{xy,2}^{\lambda}(z') &= [(1-\lambda)q(z,x) + \lambda q(z,y)] - [(1-\lambda)q(z',x) + \lambda q(z',y)]\\ &= (1-\lambda)[q(z,x) - q(z',x)] + \lambda[q(z,y) - q(z',y)]\\ &\leq q(z,z').\end{aligned}$$

Thus $\varphi_{xy}^{\lambda} \in \mathcal{A}_1(X, q)$.

Since (X, q) an Isbell-convex T_0-quasi-metric space, (X, q) is injective. Then the map $e_X : (X, q) \to \mathcal{E}(X, q)$ defined by $e_X(x) = f_x$ whenever $x \in X$, is an isometry. We consider the retraction map $p : \mathcal{A}(X, q) \to \mathcal{E}(X, q)$ in Proposition 5.2. For any $x, y \in X$ and $\lambda \in [0, 1]$, we set $\sigma(x, y, \lambda) := (e_X^{-1} \circ p) \circ \varphi_{xy}^{\lambda}$.

Now we have to show that σ is a conical geodesic bicombing on X.

On sees that σ is well defined. Observe that if $\lambda = 0$, then $\varphi^{\lambda}_{xy} = ((f_x)_1, (f_x)_2)$. Moreover, if $\lambda = 1$, then $\varphi^{\lambda}_{xy} = ((f_y)_1, (f_y)_2)$. It follows that

$$\sigma(x, y, 0) = (e_X^{-1} \circ p) \circ (((f_x)_1, (f_x)_2)) = e_X^{-1}(e_X(x)) = x$$

and

$$\sigma(x, y, 1) = (e_X^{-1} \circ p) \circ (((f_y)_1, (f_y)_2)) = e_X^{-1}(e_X(y)) = y,$$

since $((f_x)_1, (f_x)_2), ((f_y)_1, (f_y)_2) \in \mathcal{E}(X, q)$. Let $x, y \in X$ and $\lambda, \lambda' \in [0, 1]$. Then

$$\begin{aligned} q(\sigma(x, y, \lambda), \sigma(x, y, \lambda')) &= D(e_X(\sigma(x, y, \lambda)), e_X(\sigma(x, y, \lambda'))) \\ &= D[e_X(e_X^{-1}(p(\varphi^{\lambda}_{xy}))), e_X(e_X^{-1}(p(\varphi^{\lambda'}_{xy})))] \\ &= D(p(\varphi^{\lambda}_{xy}), p(\varphi^{\lambda'}_{xy})). \end{aligned}$$

Hence

$$q(\sigma(x, y, \lambda), \sigma(x, y, \lambda')) \leq D(\varphi^{\lambda}_{xy}, \varphi^{\lambda'}_{xy})$$

since p is a retraction.

Furthermore,

$$D(\varphi^{\lambda}_{xy}, \varphi^{\lambda'}_{xy}) = \sup_{z \in X} [(1-\lambda)q(x, z) + \lambda q(y, z) \dot{-} (1-\lambda')q(x, z) + \lambda' q(y, z)].$$

If $\lambda' \geq \lambda$, then by triangle inequality we have

$$D(\varphi^{\lambda}_{xy}, \varphi^{\lambda'}_{xy}) \leq (\lambda' - \lambda)q(x, y).$$

If $\lambda' < \lambda$, then by triangle inequality we have

$$D(\varphi^{\lambda}_{xy}, \varphi^{\lambda'}_{xy}) \leq (\lambda - \lambda')q(y, x).$$

It follows that if $\lambda' \geq \lambda$, then

$$q(\sigma(x, y, \lambda), \sigma(x, y, \lambda')) \leq (\lambda' - \lambda)q(x, y) \tag{5.5}$$

and if $\lambda' < \lambda$, then

$$q(\sigma(x, y, \lambda), \sigma(x, y, \lambda')) \leq (\lambda - \lambda')q(y, x). \tag{5.6}$$

Observe that for any $x, y \in X$ and $0 \leq \lambda \leq \lambda' \leq 1$, since $\sigma(x, y, 0) = x$ and $\sigma(x, y, 1) = y$ we have the following equality from the inequality (5.5)

$$q(\sigma(x, y, \lambda), \sigma(x, y, \lambda')) = (\lambda' - \lambda)q(x, y).$$

Similarly, we obtain from inequality (5.6)

$$q(\sigma(x, y, \lambda), \sigma(x, y, \lambda')) = (\lambda - \lambda')q(y, x).$$

Therefore, σ is a geodesic bicombing on X. It remains to show that σ satisfies the property (4.1) to be conical. Let $x, y, x', y' \in X$ and $\lambda \in [0, 1]$. Then

$$\begin{aligned} D(\varphi^{\lambda}_{xy}, \varphi^{\lambda}_{x'y'}) &= \sup_{z \in X} [(1-\lambda)q(x, z) + \lambda q(y, z) \dot{-} (1-\lambda)q(x', z) + \lambda q(y', z)] \\ &\leq (1-\lambda)q(x, x') + \lambda q(y, y'). \end{aligned}$$

Hence

$$q(\sigma(x,y,\lambda),\sigma(x',y',\lambda) \leq D(\varphi^{\lambda}_{xy},\varphi^{\lambda}_{x'y'}) \leq (1-\lambda)q(x,x') + \lambda q(y,y').$$

Thus σ is a conical geodesic bicombing on X.

The equivariance follows from the observations below: for $z \in X$, we have

$$\varphi^{1-\lambda}_{yx,1}(z) = \lambda(f_y)_1(z) + (1-\lambda)(f_x)_1(z) = \varphi^{\lambda}_{xy,1}(z)$$

and

$$\varphi^{1-\lambda}_{yx,2}(z) = \lambda(f_y)_2(z) + (1-\lambda)(f_x)_2(z) = \varphi^{\lambda}_{xy,2}(z)$$

whenever $x, y \in X$ and $\lambda \in [0,1]$. □

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